"THE USE OF IMPLIED METHODOLOGIES IN MATHEMATICAL FINANCE"

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DISCLAIMER

TABLE OF ABBREVIATIONS

AA	Arithmetic Average
AR(p)	AutoRegressive model of order p
ARMA(p,q)	AutoRegressive moving Average model of order p,q
AV_A	Arithmetic Average
AV_G	Geometric Average
B & S	Black & Scholes
BAW	Barone-Adesi-Whaley
BIS	Bank of International Settlements
CEV	Constant Elasticity of Variance
CRR	Cox, Ross and Rubinstein
DVF	Deterministic Voaltility Function
EMS	European Monetary System
ERM	Exchange Rate Mechanism
GA	Geometric Average
GARCH	Generalised Autoregressive Conditional Heteroscedasticity
GBM	Geometric Brownian Motion
GMM	Generalised Method of Moments
LBA	Lower Bound Approximation
LIFFE	London International Financial Futures Exchange
LUBA	Lower Upper Bound Approximation
MC	Monte Carlo
Oct	October
OP	Option
OTC	Over The Counter
PDF	Probability Density Function
PHLX	Philadelphia Exchange
RHS	Right Hand Side
RND	Risk Neutral Density
S&P	Standard & Poor's
SDE	Stochastic Differential Equation
Sep	September
SV	Stochastic Volatility
SVJ	Stochastic Volatility and Stochastic Jumps
SVSI	Stochastic Volatility and Stochastic Interest Rates
T-Bill	Treasury Bill
UK	United Kingdom
US	United States

TABLE OF SYMBOLS

υ_s	Partial Derivative of υ with respect to S
ΔP	Difference operator on P
f'	First derivative of f
dS	Infinitesimal Change in S
$\sum\limits_{i=k}^{n}$	Summation Operator. The variable i takes values from k to n
$\prod\limits_{i=k}^{n}$	Product Operator. The variable i takes values from k to n
$\int\limits_{a}^{b}$	Integration Operator.
$\frac{\partial^2 f}{\partial x^2}$	Second Partial Derivative of the Function f with respect to x
$p(S_T \geq K)$	Probability of S_T being greater than K.
$N(d_1)$	Cumulative Standard Normal Distribution of d_1
$E_t[]$	Expectation Conditional on Information up to time t
Q_d	The d percentage quartile
S_T	Value of the stochastic process S at time T
$(S-K)^+$	Maximum of S and K
$\overline{J}$	The opposite event of J
$N(a,b)$	Normal Distribution of mean a and variance b
$\equiv$	Equal by Identity
$\Phi[\ ]$	Cumulative Standard Normal Distribution

$n!$	n factorial $(= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)$
$\mathrm{Exp}(a)$	e^{a}
$Var(x)$	The variance of x

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ABSTRACT

This thesis has as an objective to explore the uses of implied methodologies in the area of Mathematical Finance. The existing literature broadly separates the ways that implied methodologies can be exploited in to two different categories; for purposes of recovery of the market sentiment and for consistent pricing of exotic and derivatives.

I explore and exploit the first use by examining the possibility of using an implied distribution of a mixture of two lognormal distributions in order to predict the macroeconomic event of the sterling pound's exit from the ERM in 1992. Market evidence presented, indicate that the market was indeed expecting a sterling devaluation a few days prior to the exit. Furthermore, the two component lognormal mixtures distribution is proven to be a powerful tool for assessing the market sentiment, especially when there are two possible scenarios for the future movement of the underlying asset.

Subsequently, I am using a jump diffusion stochastic process with a Bernoulli distributed jump component, the parameters of which are implicitly derived from observed option prices, for the pricing of exotic options. Closed form valuation expressions are provided within this generalised approach for Asian and Basket options. Furthermore, analytical formulae for the hedging parameters of those exotic products are derived. Monte Carlo simulation confirms the validity of all the results presented in this thesis.

To my parents Vassilis and Aggela

"The Use of Implied Methodologies in Mathematical Finance"

1 CHAPTER ONE: INTRODUCTION

Derivative instruments have been in existence for a much longer time than most of us would imagine. Gastineau (1979) records one of the earliest examples of use of options as it appears in Aristotle's *Politics.* Thales the Milesian, an ancient Greek philosopher-mathematician, negotiated (during winter time when demand was very little) the use of olive presses for the following spring. Thales was trying to profit from his ability in long-range weather forecasting when he was forecasting a good olive harvest.

In the early seventeenth century options were also used in Holland during the tulip bulb craze. The first official appearance of options in the United Sates was in the 1790s during the beginnings of the New York Stock Exchange (NYSE). The concept of "Put-Call parity", originally known as conversion, was first understood during the late nineteenth century by Russel Sage, a great railroad speculator who is considered from many to be the father of modern option pricing.

Modern option pricing theory, though, has been greatly based on a paper published by Fisher Black & Myron Scholes (B&S hereafter) in 1973. In this paper B&S demonstrated that a closed form solution for the value of a European option contract could be found if certain assumptions were made about the conditions in the market. The main assumptions that they proposed were the following:

1. Markets are efficient.
2. There are no market frictions (transaction costs, etc.)
3. Interest rates are constant throughout the life of the option contract.

4. The underlying asset S follows a stochastic process that can be described by the following equation:

$$dS = \mu S dt + \sigma S dZ \tag{1}$$

where dZ is a Wiener process and μ and σ is the instantaneous return drift and volatility parameters.

Under the above-mentioned assumptions the value of the option will depend only on the price of the stock, on time and on variables that are taken to be known constants. A simple application of Ito's lemma shows that it is possible to create a hedged position consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock, but will depend only on time and the values of known constants. Writing $\upsilon(S,t)$ for the value of the option as a function of the stock price S and time t, the number of the options that must be shorted against one unit of a share long is:

$$1/\upsilon_S(S,t) \tag{2}$$

where $\upsilon_S(S,t)$ is the partial derivative of the option price with respect to the stock price.

If the stock price changes by a small amount ΔS the option price will change by an amount $\upsilon_S(S,t) \times \Delta S$, and the number of options given by expression (1) will change by an amount ΔS. Thus the change in the value of a long position in the stock should theoretically be exactly offset by the change in value of the short options' position. If the hedge is rebalanced continuously the return on the hedged position is completely independent of the change in the value of the stock. In fact, the return on the hedged position becomes deterministic. This is made evident with the following through the calculations that follow.

Assumption 4 implies that the stock price evolution over time can be described by a *stochastic differential equation* (SDE hereafter) as follows:

$$dS = \mu S dt + \sigma S dZ$$

or

$$\Delta S = \mu S \Delta t + \sigma S \Delta Z \qquad \textbf{(3)}$$

in the discrete form equivalent. In the above equations μ is the expected return of the stock, σ is the standard deviation of the stock price returns and dZ is a Wiener process.

The change in the value of the option will satisfy the SDE given by Ito's lemma:

$$d\upsilon = \left(\upsilon_S \mu S + \upsilon_t + \frac{1}{2} \upsilon_{SS} \sigma^2 S^2 \right) dt + \upsilon_S \sigma S dX$$

or

$$\Delta \upsilon = \left(\upsilon_S \mu S + \upsilon_t + \frac{1}{2} \upsilon_{SS} \sigma^2 S^2 \right) \Delta t + \upsilon_S \sigma S \Delta X \qquad \textbf{(4)}$$

The portfolio that consists of one share long and $1/\upsilon_S(S,t)$ options short is worth:

$$P = S - \frac{1}{\upsilon_S(S,t)} \upsilon(S,t) \qquad \textbf{(5)}$$

and the change in its value is:

$$\Delta P = \Delta S - \frac{1}{\upsilon_S(S,t)}\Delta \upsilon(S,t) \tag{6}$$

and by substituting (3) and (4) in (6):

$$\Delta P = \mu S \Delta t + \sigma S \Delta X - \frac{1}{\upsilon_S}\left(\left(\upsilon_S \mu S + \upsilon_t + \frac{1}{2}\upsilon_{SS}\sigma^2 S^2\right)\Delta t + \upsilon_S \sigma S \Delta X\right)$$

$$\Delta P = \frac{1}{\upsilon_S}\left(-\upsilon_t - \frac{1}{2}\upsilon_{SS}\sigma^2 S^2\right)\Delta t \tag{7}$$

Since the portfolio is riskless, in the absence of arbitrage opportunities, it should yield the risk free interest rate r. Therefore:

$$\Delta P = rP\Delta t \tag{8}$$

By substituting and dropping out Δt we get:

$$\frac{1}{\upsilon_S}\left(-\upsilon_t - \frac{1}{2}\upsilon_{SS}\sigma^2 S^2\right) = rP$$

$$\frac{1}{\upsilon_S}\left(-\upsilon_t - \frac{1}{2}\upsilon_{SS}\sigma^2 S^2\right) = r\left[S - \frac{1}{\upsilon_S}\upsilon(S,t)\right]$$

$$r\upsilon(S,t) = \upsilon_t + rS\upsilon_S + \frac{1}{2}\upsilon_{SS}\sigma^2 S^2 \tag{9}$$

The significance of equation (9) lies to the fact that μ, which varies from investor to investor, is eliminated, thus creating a preference-free position. As a result the option can be priced irrespective of investors' views towards risk. The resulting differential equation can be solved subject to the boundary conditions that the nature of the option each time implies. In fact it can be transformed to a differential equation known as the heat-transfer equation in physics. In the simplest case of a European call option the solution is the following:

$$\upsilon(S,t) = SN(d_1) - Xe^{r(T-t)}N(d_2) \tag{10}$$

where

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and $N(\cdot)$ is the cumulative normal density function.

A few years later Fischer Black (1976) developed a formula for the pricing of commodity contracts. The formula serves as an extension to the original Black-Scholes formulas and it is derived under the same framework. A key assumption to the derivation of Black's formula is that the expected return on a futures contract is zero, since no initial capital needs to be engaged to enter the position. A riskless hedge can be created, as in the case of the original model, by taking a long position in the option and a short position in the futures contract with the same transaction date. Since the value of a futures contract is always zero, the equity in this position is just the value of the option. The Black (1976) formula is very similar to the one presented in equation (9) and looks as follows:

$$\upsilon(F,t) = e^{-r(T-t)}[FN(d_1) - XN(d_2)] \tag{11}$$

$$d_1 = \frac{\ln(F/X) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(F / X) - \frac{1}{2}\sigma^2 (T - t)}{\sigma\sqrt{T - t}}$$

However, throughout the years the B&S formula has been proven to have some deficiencies in describing real option markets. Empirical studies by MacBeth and Merville (1980), Rubinstein (1985) and Whaley (1982) show these deficiencies. Similar results were found also in other underlying markets. For example Bodurtha and Courtadon (1987) showed that the modified B&S formula developed by Garman and Cohlagen (1983) exhibits systematic mispricing of foreign currency options in a study conducted for the PHLX exchange. Hsieh and Manas-Anton (1998) found similar results in the DM futures option market on a study conducted on the Chicago Mercantile Exchange concerning Black's (1976) formula for the valuation of options on futures contracts.

The mispricings that the B&S formula and its variations exhibit were attributed to failure of its restrictive assumptions. Any of the four main assumptions previously mentioned could fail causing the B&S formula to give a wrong value to the option. In the literature researchers have developed models to relax mainly the last three assumptions.

2 CHAPTER TWO: LITERATURE REVIEW

Academic researchers and practitioners, have directed their efforts in relaxing the restrictive assumptions that underlie the Black & Scholes model. The first assumption was shown to be the strongest one since most of the published articles in the financial journals did not try to dispute it. However, the rest of the assumptions were attacked by a large number of papers.

2.1 The Constant Volatility assumption

The volatility entering the B&S formula is the only unobserved parameter since we do have estimates for all the other parameters. The spot price of the underlying S_0, is observed from the market, we know X, the strike price, the time to maturity T is directly calculated and we can have a very good idea about the prevailing interest rate r by looking at the T-Bill maturing closest to the option expiration. One measure for volatility is the historical estimate, or in other words, the time series volatility. Another measure for volatility of the underlying asset that has been proposed is the implied volatility. This is the value of the volatility σ that if it is inserted in formula (1) then it will yield the observed option price.

According to the B&S model, the implied volatilities across a set of options with different strike prices, and with all the other characteristics identical, should all be the same. However what is observed in the market is that the implied volatilities differ across such an option set. If the options are written on a stock, or a stock index, then for data after the 1987 crash, it has been found that implied volatilities tend to be higher for out-of-the-money puts (in-the-money calls) and lower for in-the-money puts (out-of-the-money calls), than the B&S model would predict, Rubinstein (1985). This stylised fact is called the volatility smile because of the image of the graph of implied volatility against strike price.

There is mixed evidence concerning the role of implied volatility as a forecasting tool for future standard deviation. For example Jorion (1995) finds that implied standard deviations are outperforming historical time-series measures in a foreign exchange study. On the other hand Canina and Figlewski (1993) , in an S&P 100 index options study, find that implied volatility has virtually no correlation with future realised volatility and that it does not incorporate the information contained in recently observed volatility. What is certain however, is that the assumption of constant volatility in the B&S framework is a false one.

Alternative models have been applied to take into account the fact that volatility changes during the lifetime of the option. An econometric approach has been developed by Engle (1982) who assumed that volatility follows an autoregressive of order p AR(p) (1982) or an autoregressive moving average of order p and q ARMA(p,q) (1986) time series process. This approach has been expanded, and is being expanded, in many ways by defining alternative specifications for the assumed time-series model that the volatility series follows. Alternative approaches were generated in order to take into account some of the empirical facts observed in the markets. Volatility may depend on the level of the underlying asset which gave rise to the GARCH in mean model and market participants react differently to good and bad news which motivated researchers to develop the asymmetric GARCH model.

Researchers have also modelled volatility as if it were behaving in a random way, building models of stochastic volatility. Hull and White (1987), Wiggins (1987), Melino and Turnbull (1990) and Scott (1987) have made the assumption that the underlying asset's dynamics can be described by the following equations:

$$\begin{aligned} dS &= \mu S dt + \sigma S dZ_1 \\ d\sigma &= f(\sigma)dt + \theta\sigma dZ_2 \end{aligned} \qquad \textbf{(12)}$$

where dZ_1 and dZ_2 are two Wiener processes with a correlation coefficient $\rho dt = dZ_1 dZ_2$.

These papers have the disadvantage that no close form solution can be derived for the price of a vanilla option and the use of extensive numerical techniques is required to solve two-dimensional partial differential equations. Stein and Stein (1991) have defined the function $f(\sigma)$ to be equal to $-\delta(\sigma-\theta)$ and making volatility follow an Ornstein-Uhlenbeck (or AR1) mean reverting process. Their process is given by:

$$\begin{aligned} dS &= \mu S dt + \sigma S dZ_1 \\ d\sigma &= -\delta(\sigma - \kappa)dt + \theta dZ_2 \end{aligned}$$

,where dZ_1 and dZ_2 are uncorrelated. Due to this specification the model cannot capture important skewness effects that that arise from such correlation. Heston (1993), also used the Ornstein-Uhlenbeck specification in Stein and Stein (1991) but provided a closed form solution by making an assumption for the price of the volatility risk and by guessing a functional form for the option price.

All these papers tried to describe new specifications for the process that the underlying asset's volatility could follow. The main component of the stochastic process of the underlying asset remained the Geometric Brownian Motion (GBM hereafter).

2.2 The GBM assumption

The constant volatility assumption is not the only reason that B&S option price estimates deviate, to a significant extent in some cases, from market observed prices. The assumption described by equation (1), confines also the description of reality within the B&S framework to a small class of models. Equation (1) is a Stochastic Differential Equation that has as a solution the lognormal distribution. This means that if it is assumed that the stochastic process that governs the movement of the underlying asset's price is described by equation (1) then the distribution of the underlying asset's price belongs to the family of lognormal distributions. In financial terms this can be translated as saying that the distribution of prices is *lognormal,* or equivalently that the distribution of returns is *normal.*

It can be easily shown, however, that the distribution of returns is not always normal. Moreover, after the 1987 crash, stock return distributions seem to exhibit fatter tails towards the left side of the distribution than the symmetric normal distribution does[1], giving relatively more weight to the probability of future downwards underlying movements. An explanation for this empirical fact is the inverse relationship that exists between the level of the underlying equity prices and the conditional volatility of the underlying returns, observed by Black (1976) for individual stocks and by Nelson (1991) for broad market indices. When asset prices fall then volatility increases and the probability of even further downward movements becomes greater. This has as an effect that the returns' distribution have higher probabilities in the area where the downward movements of the underlying asset correspond to, or in other words, left fat tails are observed.

Left tails present in returns' distribution can also be interpreted in a more intuitively way, as an indication of a large probability for events generating significantly negative returns, a fear constantly present in investors' beliefs after the 1987 crash (Bates (1991)).

There are various ways to relax the assumption that the stochastic process that describes the law of motion of the underlying asset's returns is a Geometric Brownian Motion. A different specification can be given to the stochastic process that governs the movement of the underlying asset. A very popular model is the one proposed by Merton (1976). It is a jump diffusion model for the stochastic process. The dynamics of the underlying can be described by the following SDE:

$$dS = (a - \lambda k)Sdt + \sigma SdZ + dq \tag{13}$$

,where a is the instantaneous expected return on the stock, σ is the instantaneous volatility of the returns. There are two sources of uncertainty present in this equation. The first is coming from the dZ term, a Wiener process, representing the risk from news coming randomly in the market. The second source of uncertainty originates from the dq term. This is an independent Poisson process that is used to describe the arrival of important pieces of information about the underlying, or shocks, and it has the following properties:

[1] MacBeth and Merville (1980), Rubinstein (1985)

1. The probability that a jump occurs in the infinitesimally small time interval $(t,t+h)$ is equal to $\lambda h+O(h)$.

2. The probability that a jump does not occur in the time interval $(t,t+h)$ is equal to $1-\lambda h+O(h)$.

3. The probability that more than one jump occurs in the time interval $(t,t+h)$ is equal to $O(h)$.

4. We can see from the above that the B&S SDE (1) is a special case of (13) for $\lambda=0$.

The increased descriptive ability of this model comes at the cost that the jump component dq represents a non-diversifiable source of risk. This implies that the option value cannot be computed by no-arbitrage arguments. Merton has dealt with this deficit by assuming that the jump risk is uncorrelated with the market and therefore non-systematic. Nevertheless "...one cannot "act as if" the jump component was not there and compute the correct option price" as it is stated in Merton (1976). The jump risk is present and it will affect the equilibrium price.

Merton's model has accepted many modifications from late researchers. Bates (1991) derives a model to price American options with a jump-diffusion model with asymmetric jump risk. The expected percentage jump size is allowed to be non-zero as opposed to Merton's assumptions. A positive expected jump size implies that the distribution of the underlying is positively skewed while a negative expected jump size implies the exact opposite result. Bates went even further and introduced (1996b) a jump-diffusion stochastic volatility model generalising this class of models. Within this class of models he also proposed a way to compute American option prices, which is similar in spirit to the Barone-Adesi and Whaley (BAW) approximation that is valid for the Geometric Brownian Motion framework.

The same model family was used also from Bakshi, Cao and Chen (1997) who performed empirical tests of four different models. In this paper the authors tested the B&S model against three other models. One with stochastic volatility (SV), one with stochastic volatility and stochastic interest rates (SVSI), relaxing assumption 3 as well, and one with stochastic volatility and stochastic jumps (SVJ)[2]. Their findings are very useful because they give us a way to test these models and finally decide whether the additional complexity introduced by the extra formulations of the underlying asset's stochastic process is of any practical importance.

Their empirical evidence indicate that all three models have an improved pricing performance relative to the B&S formula with the stochastic volatility demonstrating the first order importance. All three models were significantly mis-specified even though they performed better than the B&S. In terms of forecasting power the SVJ model enhances the performance of the SV model for the pricing of short-term option while the SVSI model helps improve the performance of the SV model for the pricing of long-term options.

This is an intuitively expected result since changing interest rates should become important in the long run, because interest rates is a variable that does not change dramatically in short time intervals. Consequently the assumption of constant interest rates for short maturity option does not affect greatly the performance of the model while it becomes fairly restrictive for longer maturity options. On the other hand the jump component is important when modelling short maturity options because in a short time interval not many jumps will occur, and those that will, will affect significantly the price of the option. If the option matures after a longer time period then the number of jumps will increase during this period. However, jumps can be either of a positive or of a negative sign and in the long run they will tend to offset each other, in a symmetric jump model. As a result of that the price of a long maturity option in the absence of a jump component will not be greatly mis-priced relatively to that coming from a jump model.

For hedging purposes incorporating either the jump feature or the stochastic jump feature did not improve the hedging performance of the SV model, since the more parsimonious SV model had the best hedging results. This indicates that the benefits coming from a mathematical model used for option pricing are not in proportion with the complexity of the model. Dumas, Fleming and Whaley (1999) reached a similar conclusion in a study for implied models, as it will be shown at a later paragraph.

2.3 Discrete time models

Another implicit and yet again restrictive assumption of the Black & Scholes model is that trading takes place in a continuous fashion. This is not a valid description of the trading environment, though, since trading even in the most liquid markets does take place in discrete time intervals. Furthermore, discrete-time models are easier applicable on a computer since all the mathematical packages and the

[2] Models with stochastic interest rates have been developed by Merton (1973) and Amin and Jarrow (1992). Models with stochastic volatility and stochastic interest rates have been developed by Amin and Ng (1987) Bailey and Stulz (1989), Bakshi and Chen (1997b).

software used by researchers can only deal with finite small numbers, numbers of finite precision, and not infinitesimally small, as continuous-time models require.

Among the discrete-time models the most celebrated one is the binomial tree model that was first developed by Sharpe (1978). However, discrete-time models were never "autonomous". They were always built in order to match some continuous time theory. As their name suggests models belonging to this class are the result of the discretisation of continuous time theory models against which they were calibrated. That also implies that since discrete time models are an approximation of the limiting continuous time theory, they are also bound to approximation errors.

Cox, Ross and Rubinstein (1979), CRR hereafter, developed a discrete time model and showed that a suitably defined binomial model for the evolution of the stock price converges weakly to a lognormal diffusion equation as the time between consecutive time steps shrinks towards zero. They demonstrated in this case that the call value converges to the value given by the B&S formula. They made explicit the connection between the continuous time valuation equation (which is the fundamental partial differential equation for the contingent claim) and the discrete-time one-period valuation formula developed under the assumption of a binomial model for stock prices.

Due to the importance of the model in the mathematical finance area as a subsequent research topic and its wide usage in the exotic option valuation as well as in the implied model literature, it will be briefly presented and analysed in this section.

2.3.1 Tree Construction

The economy examined by CRR has only one time period between present time and expiration. Furthermore, only three assets exist: a riskless zero coupon bond B, paying 1 currency unit at maturity, a stock, S that pays no dividends, and a derivative security, say a call option C. There are assumed to be only two states of the world at maturity time, since the asset is only allowed to go up to a certain up-value or go down to a certain down-value.

The framework will become more clear with a numerical example.

2.3.1.1

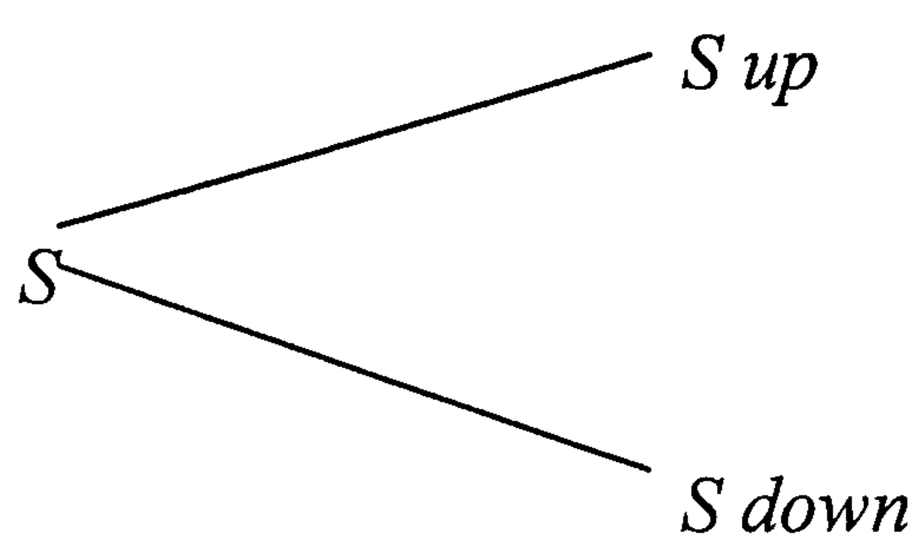

Figure 1: A two state one-period model for the stock price

Assume that the call option is also added to the model. The one-period model with the addition of the cal price is now changed to:

2.3.1.2

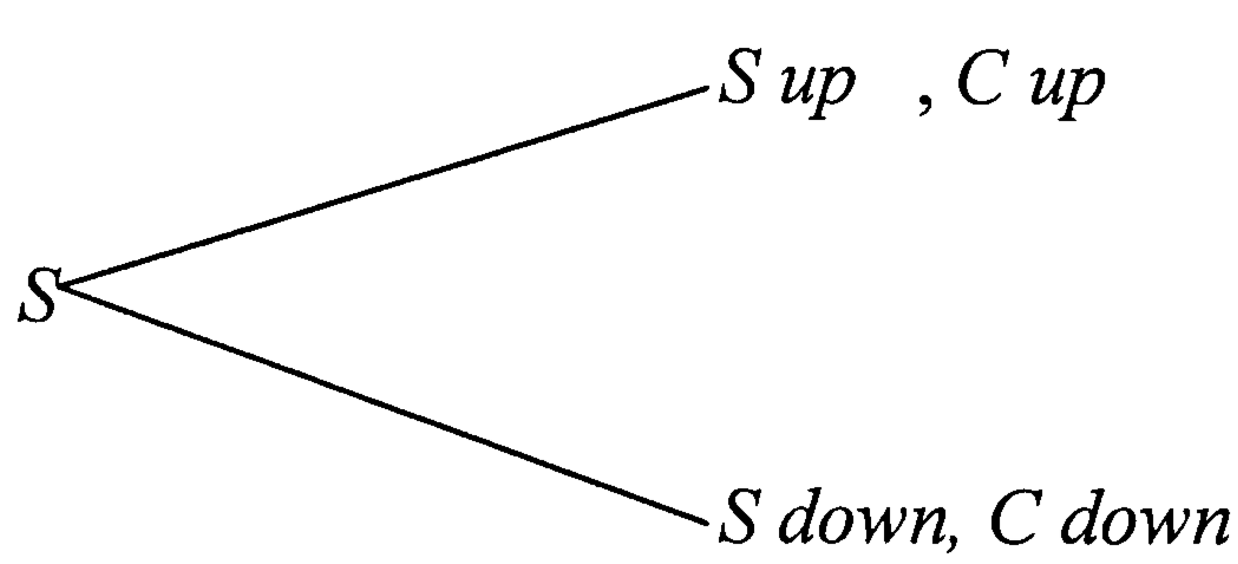

Figure 2: The two state one period model including the derivative asset.

The only assumption that needs to be made in order to price this option is that there are no-arbitrage opportunities. A portfolio is set up consisting of a long position in an amount of Δ shares and a short position in one unit of the call option. So far the riskless bond has not been used. We will try to determine the value of Δ that will make the portfolio have the exact same payoff as the riskless bond. Since there are only two states in the world and two securities then there is always the possibility to find a value for Δ that will render the portfolio riskless.

In the up-state the portfolio is worth

$$\Pi = \Delta Sup - C = \Delta Sup - Cup$$

In the down-state the portfolio is worth

$$\Pi = \Delta Sdown - Cdown$$

If the portfolio is riskless then the payoff must be the same in all states. Equating the two equations it can be found that the amount of shares held should be:

$$\Delta = \frac{Cup - Cdown}{Sup - Sdown} \tag{14}$$

Expression (14) shows that for this specific value of Δ the portfolio will be riskless, which implies that it should earn the risk-free interest rate. Its value at maturity will be equal to $\Delta Sup - Cup$ independently of the state. Discounting this value to present time we get that the value of the portfolio now should be equal to:

$$\Delta S - C = e^{-r(T-t_0)}(\Delta Sup - Cup)$$

Or in other words the value of the option today is equal to:

$$C = \Delta S - e^{-r(T-t_0)}(\Delta Sup - Cup) \tag{15}$$

This shows that in the absence of arbitrage opportunities it is possible to value a European call option within the simple binomial framework. CRR further assumed that *Sup* and *Sdown* are given by the following equations:

$$Sup = S \cdot u$$
$$Sdown = S \cdot d$$

with $u = 1/d$

Replacing these values in (14) and substituting the value of Δ we derive the following expression:

$$C = e^{-r(T-t_0)}[pCup + (1-p)Cdown]$$

where

$$p = \frac{e^{r(t-t_0)} - d}{u-d}$$

(16)

Can the variable p be interpreted as the probability of an up movement of the underlying asset? If that is the case then the expected value of the underlying is given by:

$$E(S_T) = pSu + (1-p)Sd$$

Substituting the value of p from

(16) then we find that:

$$E(S_T) = Se^{r(T-t_0)}$$

This equation shows that if we are to interpret the variable p as a valid probability then according to this probability the underlying asset is expected to grow at the risk-free interest rate. Because of this result p is called the risk-neutral probability.

The methodology can easily be extended to the general case of n-time steps in a multi-period setting. At each time step j there are $j+1$ states of nature and at each node of the tree the value of S is given by:

$$S_{i,j} = S \cdot d^i u^{2j}$$

In the above equation i ranges from 0 to n and it shows at which time step we are and j ranges from 0 for the bottom node until i for the top one. A four-period tree and the stock price values at each node are depicted in the figure depicted in the next page.

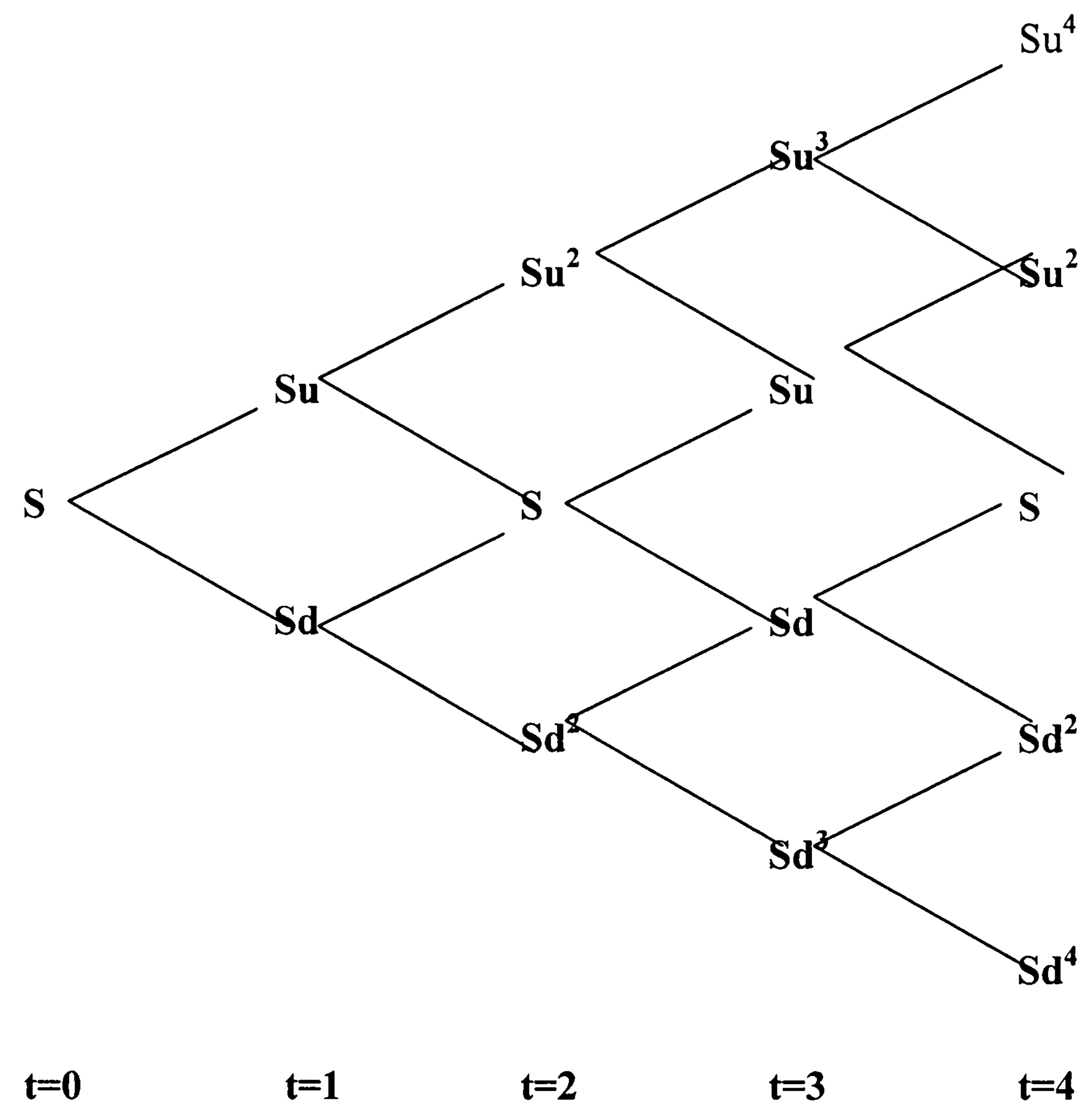

Figure 3: Multiperiod recombining binomial tree.

The binomial model has enabled traders to value contingent claims in environments where a closed form solution was unavailable. American options and other exotic derivatives with complex payoffs can be valued in a relatively straightforward manner on a binomial tree. To demonstrate how this is feasible, for American options for example, it is possible to calculate the value of such an option using the exact same algorithm that is used for European options with the sole modification that at each node the European value of the option must be compared with the value of early exercise at that node and the value of the American option at that node be set equal to the maximum of these two quantities.

As it was mentioned earlier, discrete time models are made in order to match a continuous time theory. Cox, Ross and Rubinstein (1979) and also Brennan and

Schwartz (1978), proved that the solution to the option valuation problem though a binomial tree is equivalent to a numerical solution to the exact same partial differential equation that Black and Scholes derived and solved. The parameters of the binomial tree are determined from the stochastic process that the underlying asset is assumed to follow. There are two parameters in a binomial tree the u and the d; the percentage by which the underlying asset goes up or down at its next move. The value of d is however restricted to be equal to the reciprocal of u, a condition set to guarantee that the tree will recombine. This implies that if the underlying asset moves first up and then down or first down and then up then it will end up in the same asset value. The remaining degree of freedom, the value of u, is determined in such a way so that the limiting case of the binomial tree as the number of intermediate time steps goes to infinity, be consistent with the Geometric Brownian Motion assumption of the Black and Scholes model.

Cox, Ross and Rubinstein (1979) found that if the underlying stock price dynamics are given by the GBM formula:

$$dS_t = rS_t dt + \sigma S_t dZ_t$$

then the value of u should be calculated by the following equation[3]:

$$u = e^{\sigma\sqrt{\Delta t}}$$

The convergence of the CRR binomial method to the B&S formula means that the approach does have all the drawbacks that the B&S framework presents. This implies that the initial form of the binomial method although it provided a flexible valuation environment not to mention a very pedagogical one, it was still bound to the same limitations of the B&S assumptions. There was a need for a change in the binomial method assumptions so that the model could incorporate more complex specifications for the stochastic process for the underlying. Cox and Rubinstein (1985, p.362) have applied the binomial approximation to the constant elasticity of variance (CEV) model but it turned out that the number of mathematical operations needed for the evaluation of an option through the tree, grow exponentially with the number of time-steps, making it a non-practical model for real time application. Nelson and Ramaswamy (1990) show how to construct binomial processes that converge weakly to commonly employed diffusions in financial models. Numerical examples for the CEV process:

$$dS = \mu S dt + \sigma S^{\gamma} dZ \qquad \text{with } 0 < \gamma \leq 1$$

and for the Cox, Ingersoll and Ross autoregressive square root interest rate process (1979):

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dZ \qquad \text{with } \kappa \geq 0, \mu \geq 0$$

were given in the paper. Buchen (2000) also suggests a method for the construction of generalised binomial trees for arbitrary Ito processes.

Most of the processes though, that have been described earlier[4], have not been discretised using a binomial method. Researchers had to proceed in a different way to relax the restricting B&S assumptions. The answer was in the use of implied methodologies, which are more informative on the risk-neutral distribution of the underlying asset or the stochastic process that it follows. Research in the area of

[3] For more details see Cox , Ross and Runbinstein (1979)

[4] Processes that correspond to the jump diffusion model or to the stochastic volatility model.

implied binomial trees was motivated by the work of Francis Longstaff (1990) published in (1995). The paper by Longstaff was the first paper conducting research in the implied models area, since Breeden and Litzenberger (1978), inspiring modern scientists to pursue further research in this area.

Longstaff's paper was followed by that of David Shimko (1991) and (1993), which will be analysed later in the text and three other papers that were published almost simultaneously in 1994 by Rubinstein, Dupire as well as Derman and Kani. All of them had as a topic implied binomial or trinomial trees and all of them tried to find a pricing model that accounted for the empirical fact of the volatility smile.

Rubinstein in his paper constructed an implied binomial tree in such a way so that the terminal risk neutral distribution of the underlying asset, resulting from the tree, be as close as possible, in the least squares sense, to an *a priori* given distribution. The choice he made for this distribution was the lognormal distribution for the stock price, due to its pre-eminence in the financial world. However, this method can be applied for any choice for an ending risk-neutral distribution. The market's sentiment about the RND function for some date in the future can be assessed, allowing us to estimate the risk neutral expected value of the underlying asset but more importantly the shape of the probability distribution and the degree of skewness and kurtosis.

The implied distributions recovered by the implied binomial tree as proposed by Rubisntein (1994) exhibited a "bumpy" behaviour. An improved version of the tree was presented in Jackwerth and Rubinstein (1996), with more emphasis given to the choice of the objective function. The authors wished to create a procedure that would insure that the recovered distribution would not display the erratic behaviour previously observed. To accomplish that they added one more constraint to the optimisation procedure involved in the recovery of the implied probabilities from the observed option prices. Namely, they selected the set of implied end probabilities $\{P_i\}$ that minimise the following function.

$$f = \sum_{j=0}^{n+1} (P_{j-1} - 2P_j + P_{j+1})^2 \qquad \text{where } P_{-1} = P_{n+1} = 0$$

The formula in the parenthesis that is being minimised is the discrete analogue of the second derivative of the probability distribution function. Minimising this quantity insures that the set of probabilities recovered will be the smoothest possible candidates. The tree is then constructed in a backward fashion. Having the ending risk neutral probabilities at hand the nodal probabilities for the rest of the tree are calculated through a very simple procedure ("as simple as one-two-three", as Rubinstein characteristically emphasises in his paper).

Rubinstein's method extracts the risk-neutral probabilities from market data, for a specific date in the future, a date that coincides with the expiration date of the traded option contracts used in the tree calibration. Consequently, Rubinstein's (1994) or Jackwerth and Rubinstein's (1996) method utilises information only from options maturing at a specific date in the future across all available strikes.

Dupire (1994) followed a different approach concentrating on the stochastic process that the underlying asset follows. The assumption that he suggested was that the underlying asset evolves according to a stochastic process that satisfies the following SDE:

$$dS_t = a(S_t, t)dt + b(S_t, t)dZ_t$$

where for simplicity, he assumed that $a(S_t,t)$ was equal to zero. He demonstrated a way to infer $b(S_t,t)$ from observed option prices and a way to build a trinomial tree consistent with the retrieved stochastic process. Assuming that $b(S_t,t)$ is a more general function of S_t and time, is a generalisation of the B&S framework since the latter is a special case of this model, or in other words it is nested in this model for the choice of $b(S_t,t)=\sigma S_t$ and $a(S_t,t)=(r-d)S_t$. The tree is constructed in a forward fashion, and the nodes at each time step are determined by option prices expiring at that time step. For this reason, though, the method employs heavily interpolation and extrapolation techniques, in order to produce the necessary spectrum of option prices. Dupire is not attempting to recover a functional form for $b(S_t,t)$ but rather to estimate it numerically. His approach is important because it aimed to model the underlying asset's dynamics, and could thus be used as a possible hedging tool. Rubistein's method on the other hand, cannot be used for such purposes since there are no assumptions made about the stochastic process governing the movement of the underlying asset. Furthermore, since a terminal distribution is consistent with an arbitrary number of stochastic processes we cannot infer the implied stochastic process by making an assumption about the terminal distribution of the asset alone.

2.3.2 Arrow-Debreu Pricing

Both of the methods described previously, even though they approach the implied option valuation problem within a discrete time framework, yet at the same time in a very different way, they have a common limitation. They can only be utilised to extract information embedded in European–style options only. This limitation stems from the way that the option payoff can be expressed in terms of certain economic quantities, called Arrow-Debreu securities. The Arrow-Debreu securities are securities that pay one unit of currency if a certain state of the world occurs at a specific point in the future and zero otherwise. The world is assumed to have a finite number of different states, which implies that we are working in a discrete time environment. Binomial or trinomial trees provide a discrete environment, which is ideal for the employment of Arrow-Debreu securities.

To understand what Arrow-Debreu securities are, one should remember that each node of a tree represents two quantities: 1) a moment in time, and 2) a possible stock price level at that point time[5]. The Arrow-Debreu price of a node $Q_{i,j}$ is the value of a security that pays £1 if the stock reaches that node (i,j) and zero otherwise.

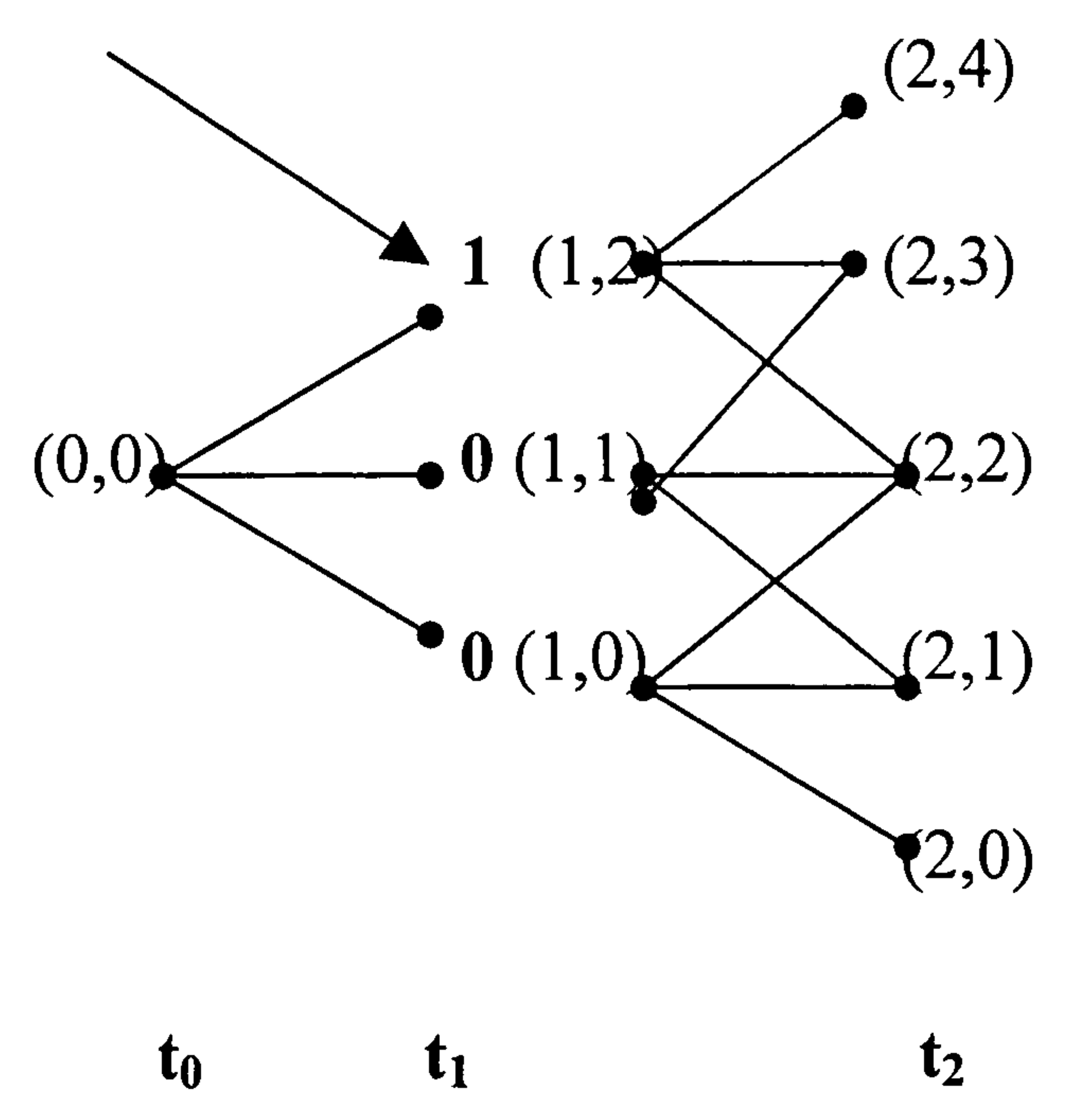

Figure 4 Trinomial tree representation

For example in the above Figure the Arrow-Debreu price $Q_{1,2}$ is the price of a security that pays £1 if state 2 occurs at time 1 and zero everywhere else.

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[5] This is valid for Rubinstein's method. In Dupire's method every node represents an option with strike price equal to the level of the tree at the specific node expiring at the specific time-step. Nevertheless the logic of using the Arrow-Debreu prices is exactly the same.

The way Arrow-Debreu securities are connected with the option valuation problem and the fact that they are indeed intuitively very useful can be explicitly demonstrated by stating an example using the previous figure.

Since a discrete time environment is being employed as a framework, with only two periods between now and maturity, we assume that there five possible states of the world at time period two. It must be noted here that in a trinomial tree there are *2n+1* nodes at the n^{th} time-step in contrast with the just *n+1* nodes in a binomial model. The crucial observation that must be made is that the value of a European-style option can be expressed in terms of the prices of the Arrow-Debreu securities in the following manner:

$$C = \sum Q_{i,T} Payoff(i,T) \tag{17}$$

with

$$Payoff(i,T) = \max(S_{T,i} - K, 0)$$

for a call option and

$$Payoff(i,T) = \max(K - S_{T,i}, 0)$$

for a put option.

In this way it possible to break down the payoff of a call or a put option expressing it as a linear combination of the constituent payoffs at each state. Equation (17) could be written because the value of a European-style option depends only on the final value of the asset and not on any intermediate values. There is no similar expression enabling the estimation of values for American-style options because equation (17) cannot take into account the early exercise feature of American options.

Furthermore, it is not possible to write an expression similar to (17) in order to express the value of more complex derivatives, the payoff of which depends on the path followed by the underlying asset during the life of the option. The use of Arrow-Debreu securities, although very useful and intuitive in many aspects, confines investors in exploiting information from European options only. The two previously described methods use these securities in order to retrieve the implied tree transition probabilities (in Rubinstein's case) or to calculate the tree connections (in Dupire's case).

Derman and Kani (1994) have introduced implied binomial tree specification, which extends the capabilities of the method proposed by Rubinstein (1994) since it utilises a wider source of information than the latter. In the first version of their method they proposed a way to build an implied binomial tree exploiting information contained in European option prices but in all traded contracts. Options across all strikes and for all traded maturities were used in order to calibrate this tree. This method appears to be "superior" to the one suggested by Rubinstein (1994) since more information coming form the market can be exploited, allowing the trader to build a tree that it is consistent with the whole of the market rather than only with options of a specific maturity. The objective of the procedure described by Derman and Kani (1994), is to construct a tree that is consistent with the observed option prices, enabling the pricing of an over the counter derivative (OTC) or of an exotic product that can be hedged with the instruments used to construct the tree. Using the same model to price and hedge the tree protects the trader from model inconsistency, an important feature of implied methodologies.

The key assumption, on which Derman and Kani[6] based their approach was that the stochastic process governing the movement of the underlying asset's price is given by the formula:

$$dS = \mu S dt + \sigma(S,t) dZ$$

This implies that volatility is a deterministic function of the stock price and time. There is only one source of uncertainty in the market and consequently markets are complete. They then deduce the unknown function $\sigma(S,t)$ numerically from the observed smile by requiring that option prices calculated from the model fit the ones observed in the market. The tree is forward-constructed and at each new time step the nodes and their implied probabilities are calculated using information from options expiring at that time step. Since there are not as many traded maturities as intermediate time steps on a tree, the intermediate option prices are generated through interpolation and extrapolation of the existing options' set. This fact made this implied tree model sensitive to the interpolation technique used which is not a desirable feature for any model.

In addition the Derman and Kani tree, turned out to be very unstable when it was being implemented. Due to inherit inefficiencies of the algorithm bad probabilities (probabilities higher than one or negative probabilities) were generated in many nodes of the tree, giving rise to possible arbitrage opportunities. The problem was dealt with by replacing the bad probabilities with valid probabilities between zero and one in an ad hoc fashion.

[6] The same authors have published a lot of papers dealing with local volatility surfaces where they assume that market variation is a function of the level of the market and of time. In this way they introduced a new concept, that of local volatility. In an implied tree this is the volatility associated with each node of the tree. See Derman, Kani and Zou (1996), Derman, Kani and Kamal (1997). For

In order to overcome these shortcomings Derman, Kani and Chriss (1996), generalised the Derman and Kani (1994) method in a trinomial tree environment. Working in such an environment is advantageous relative to working in a binomial tree one because there are more degrees of freedom (six instead of four), that can be used in the tree calibration procedure which results to a more flexible tree that provides a better fit to observed option prices.

The Derman and Kani method as proposed in 1994 could also not take into account the early exercise feature of American options and consequently it could not be used to extract information from this type of options. An improved version of this method was presented in Chriss' book "*Black-Scholes and Beyond*" where an iterative procedure was proposed in order to solve the problem of extracting implied information from American options using the Derman and Kani tree.

Barle and Cakici, (1995), (1998) improved the Derman and Kani (1994) methodology by suggesting a few modifications. In these two articles they proposed a slightly alternative way to calibrate the tree, which did result in a more stable tree. The tree would still exhibit arbitrage violations but much less frequently than the original Derman and Kani (1994) model.

Jackwerth (1997) generalised the implied binomial tree proposed by Rubinstein (1994) through the introduction of a simple weight function. The tree is constructed in a backward fashion in contrast with the Dupire (1994) and Derman and Kani (1994) models. The backward construction is governed by the weight function, which has been introduced to relax the restrictive assumption of equal path probabilities, present in Rubinstein's (1994) implied tree model. The latter model is

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other work in the area see Jackson, Suli and Howison (1998/99).

nested within the generalised binomial tree suggested by Jackwerth (1997) for a special choice of a linear weight function. The generalised tree offers the flexibility of incorporating information from American options and from other exotics at the cost, though, of a non-linear optimisation procedure. One of its most important features is that it is guaranteed to be arbitrage free with nodal probabilities everywhere positive and below one.

Brown and Toft (1999), build a semi-recombining tree to generalise the Derman and Kani (1994) methodology. They derive risk-neutral densities for each date that there is traded expiration. These probability distributions are constrained to have non-negative probabilities and also probabilities that sum to one. Using a new set of traded options each time they determine the transition probabilities between two the expiration of each set. The conditional transition probabilities must be consistent with the unconditional return distributions at those two expirations. In most cases they had employ an optimisation procedure choosing the conditional distributions that minimised a weighted sum of squared deviations from a set of prior conditional distributions. The method that they proposed incorporates the CRR (1979) approach the implied binomial tree of Rubinstein (1994) and the implied trinomial tree by Derman and Kani and Chriss (1996). Despite the flexibility of this model it is difficult to be implemented since it results in a semi-recombining tree which complicates computations and reduces the speed of the application of the algorithm.

Deterministic volatility models like the ones applied by Derman and Kani (1994), Dupire (1994), Andersen and Ratcliffe (1997), Bodurtha and Jermakyan (1996) and Lagnado and Osher (1997), are attractive for several reasons. First, within the framework of these models markets are dynamically complete. Thus, derivative securities can be priced by no arbitrage arguments alone, without the need to resort to general equilibrium models and without the need to calculate risk premia or make use of utility functions. Second, deterministic volatility models are able to fit the smile exactly by calibrating the local volatility function of the underlying asset. The attraction is that it is possible to generate state price densities that are consistent with current observed option prices. However, a natural concern regarding the practical use of these approaches is that it is possible that this improved static pricing performance might come at the cost of potential overfitting of the data since these models typically require the estimation of a large number of parameters using as inputs only the observed option prices. If this were the case the possibility of using these models for risk management purposes or for pricing exotic options would be limited.

Buraschi and Jackwerth (1998), are trying to test exactly that in a comparison between deterministic versus stochastic volatility models. Using a Generalised Method of Moments (GMM) estimation procedure they find that the null hypothesis that out-of-the-money and in-the-money options are redundant securities when the at-the-money options are used to span the payoff space, is rejected. This hypothesis, interestingly enough, is rejected both before and after the 1987 crash with the rejection being sharper after 1987. Their findings provide evidence supporting stochastic volatility models where options are not redundant securities.

Deterministic volatility models, and particularly the implied methodologies of Derman and Kani (1994), of Dupire (1994) and of Rubinstein (1994) received further criticism by Dumas, Fleming and Whaley (1999). In their methodology, they offered four different parameterisations for the volatility function for these deterministic volatility function models (DVF). They found that the parameter estimates were unstable in time and that the predictive and hedging performance was not better than that of an ad hoc procedure that smoothes B&S implied volatilities, across exercise prices and times to expiration.

However, there were some problems with the way the testing procedure was set up. Concerning the data, for half of the dates used in the study, there were only two expiration dates available, making it extremely difficult to extract any parabolic form of the volatility function with respect to time since only two points of a curve were available. They also deviated from the implied tree methodology by assuming these parametric forms in the first place, since no such assumption is made by any of the three methodologies being represented by the DVF model. The testing method is like a generalised version of Shimko's method.

Furthermore, it should be made clear that implied methodologies are not used in order to price the vanilla call or put options, since the prices of these instruments are taken as given. Nor are they used to extract hedge ratios for these simple instruments. The main contribution of implied methodology techniques when it comes to pricing, is the possibility of pricing more complex, exotic derivatives on a tree consistent with the vanilla calls and puts observed in the market, that is without suffering from model inconsistency.

2.4 Modelling distributions

The models described so far had as an objective to propose alternative forms for the evolution of the stochastic process that governs the movement of the underlying asset. The 'second generation models' attempt to explain the B&S biases by modelling the distribution that the underlying asset follows.

One way to incorporate the distribution that the underlying follows is to estimate the statistical properties of the time-series data and then make a choice for a known distribution that best matches these properties[7]. A naïve way to calculate the option price using this statistically estimated distribution $p(S_T)$ would then be by arguing that since the present value of the option is equal to its expected discounted payoff, then the following formula should give its value:

$$C(S,\tau)=e^{-r(T-\tau)}\int_{-\infty}^{\infty}p(S_T)f(S_T)dS_T \tag{18}$$

This method, however, is not guaranteed to give a rise to a consistent pricing rule, in the sense that it does not prevent arbitrage. If option prices generated by this rule, were consistent then two trading strategies that had the exact same final payoff should have the same initial cost. Alternatively, riskless profits could be generated by an appropriate trading strategy. Since markets are assumed to be efficient then no arbitrage opportunities should exist and price inconsistencies should disappear as soon as they arise.

Harrison and Kreps (1979) have shown that option prices can still be expressed as the discounted expectation of their final payoff, even within the arbitrage pricing framework, by calculating the expectation over a different density $q_t(S_T)$ subject to a different probability measure. Furthermore, Harrison and Pliska (1981) show that in a market where asset price dynamics are described by stochastic processes, satisfying certain regularity conditions, the absence of arbitrage is equivalent to the existence of a probability measure Q equivalent to the true measure P, such that all discounted asset prices are Q-martingales. This implies that if one denotes by $q_t(S_T)$ the conditional density of the stock price under the measure Q given information up to time t, then the present value of a derivative asset C_t with payoff $Payoff(C_t)$ verifies the equation:

$$C_t = e^{-r(T-t)} \int_{-\infty}^{\infty} q_t(S_T) Payoff(C_t) dS_T$$

The stock price in particular is also a Q-martingale satisfying:

$$S_t = e^{-r(T-t)} \int_{-\infty}^{\infty} q_t(S_t) S_t dS_t$$

The density $q_t(S_T)$ is called the risk neutral density. The two densities are the same only in situations where the markets are complete i.e. every possible payoff can be replicated by a self-financing portfolio strategy. This is the case in the Black & Scholes world but in the general case markets are incomplete, meaning that the two densities will be different and arbitrage arguments do not enable the calculation one from the other. The risk-neutral distribution will differ from the market's subjective probability distribution if the market prices include premiums for risk and liquidity.

The no-arbitrage approach provides us with existence theorems that state that there is a density q, integrating over which, we can compute option prices. However, the uniqueness of such a density is not guaranteed. Indeed, except in very special cases like the Black-Scholes or the binomial tree model where q is determined uniquely by no arbitrage conditions, there are infinitely many densities q that satisfy no arbitrage requirements. If this is the case then the market is incomplete.

[7] For such studies in the foreign exchange market see Rogalski and Vinso (1977), Tucker and Pond

One could argue here, however, that prices are not unique either since there is a bid and an ask price and every price in between them is considered to be an arbitrage free price. This has led to an effort to express the bid/ask prices as the supremum/infimum of arbitrage free prices, the supremum/infimum being taken over all martingale measures (Eberlein & Jacod, 1997). These approaches though, give disappointing results as Eberlein & Jacod have shown. They considered the case of a purely discontinuous process and took the supremum/infimum over all martingale measures, which led them to trivial restrictions for the derivative value. For a call option, for example, the methodology postulated that the arbitrage bounds become:

$$[S_t - Ke^{-r(T-t)}, S_t].$$

These bounds are far too large compared with typical bid ask spreads and, furthermore, they can also be retrieved with mush simpler arguments (Hull 1993).

These results show that arbitrage arguments alone are not adequate in order to determine uniquely the price of an option if we assume a stochastic process for the underlying asset that is different than the Geometric Brownian Motion and further constraints need to be imposed.

One approach is to choose among state price densities q the one which verifies a certain optimisation criterion. The optimisation criterion can either correspond to the minimisation of hedging risk (Folmer & Sonderman, 1986), or to a certain trade off between the cost and accuracy of hedging (Schal, 1994). Another criterion, proposed by Folmer and Schweizer (1990) could be to choose among all the martingale measures the one that is closest to the historical probability in terms of

(1988), Westerfield (1977).

relative entropy. In any case following a procedure like the ones described above, would lead to the derivation of a unique probability density which then would be used as the state price density.

Bouchaud et al. (1995), proposed to abandon arbitrage arguments and define the price of the option as the cost of the best hedging strategy, which minimises hedging risk in a quadratic sense. This approach which is further developed in Bouchaud and Potters (1997), is not based on arbitrage pricing and although the prices obtained coincide with the arbitrage-free ones in the case where arbitrage arguments define a unique price, they may not be necessarily arbitrage free in other situations.

3 CHAPTER THREE: IMPLIED MODELS

Following the previous analysis it is easily understood that new methods were needed in order to estimate the distribution of the underlying asset and calculate arbitrage free prices, based on that distribution. Research has then been directed towards exploiting information embedded in option prices themselves.

If we assume that the functional form of the distribution function is known then the value of any contingent claim of European type can be found under the Cox-Ross (1976) assumption of risk neutrality, as:

$$C(S,\tau) = e^{-r(T-\tau)} \int_0^{\infty} q(S_T) f(S_T) dS_T \qquad \textbf{(19)}$$

,where $f(S_T)$ is the payoff function at maturity of the contingent claim and $q(S_T)$ is the risk neutral density function of the underlying asset. From formula (19) it can be inferred that option prices may contain implicit information concerning the entire risk neutral distribution. The models that have been developed, in order to exploit information originating directly from option prices, are called implied models.

A difficulty in both the interpretation and the evaluation of risk-neutral probability density functions (PDFs) derived from option prices is the impossibility of distinguishing between actual probability, in a purely statistical sense and the risk – neutral probability. A state that may have a relatively high probability in the risk-neutral world may in fact have a relatively low statistical probability of actually occurring but simply have a high valuation. For example a stock market crash may have a very low probability of occurring but a dollar in that state is very highly valued. This will be reflected in the pricing kernel that transforms statistical probability into risk-neutral probability, but from a research point of view it is impossible to disentangle the contribution of statistical probability and the contribution of relative marginal utility of different states. In the absence of a full-scale economic model in which marginal utilities under different states of nature are made explicit, one can at best use economic intuition to identify qualitatively how the risk-neutral and empirical distributions are expected to differ.

The presence of "Peso" problems also complicates the empirical evaluation of implied PDFs. A small-probability large-magnitude event can have an important influence on the shape of a PDF even if this event is not observed over a finite sample period. The absence of this rare event from the data would not necessarily invalidate the PDF. The effect of "Peso" problems is even more pronounce if the event is associated with relatively high marginal utility as might be the case with a stock-market crash, causing the risk-neutral probability to appear higher than the statistically expected probability.

In spite of these complications, though, implied models and generally the utilisation of implied information is one of the main current research topics. Non-implied models collect their information mainly from statistical properties of the time-series data of the underlying asset or from purely theoretical arguments. The innovation that implied models introduced is that information embedded in option prices is used directly without having to be filtered through the underlying asset's properties. This directness is the informational gain that implied models are offering. Furthermore, option implied risk neutral distributions are forward looking while time-series estimated distributions are based on past data. Thus they are able of incorporating a wide range of future eventualities that simply are not captured using historical data. They do not require a long historical time series in order to be estimated accurately and in addition they are able to reflect immediately a change in market sentiment. A sudden shift in beliefs due to a political announcements or economic news could be immediately captured in option prices and the implied PDF. Furthermore, they are capable of capturing the uncertainty inherent in financial markets that of "multiple scenarios". The shape of a potential distribution function will depend on market data across strike prices rather than on a mathematical function of the standard errors of an econometric regression model.

Until recently most research in this field was concentrated around implied volatility. Now[8], however, implied distributions are becoming an increasingly popular area of research.

The basis of implied distributions though, dates back in 1978 when the path breaking paper by Breeden and Litzenberger was published. In that paper in a discrete time environment, within the time-state preference framework of Arrow (1964) and Debreu (1959) they used the Arrow-Debreu state prices of butterfly spreads to find a way to retrieve the risk neutral density (RND) function of the underlying asset's distribution. They showed that the RND function is equal to the second derivative of the call function $C(X,T)$ with respect to the strike price. In equation form:

$$\frac{\partial^2 C(X,T)}{\partial X^2} = e^{-rT} q(X) \qquad \textbf{(20)}$$

,where $q(X)$ is the RND that enters equation (19) allowing us to estimate the option price.

[8] The implied distributions area has become more popular in the mid-nineties.

The Breeden and Litzenberger (1978) paper was a breakthrough in option pricing because it allowed option traders to measure the market's actual views about the possible future underlying movements since information embedded in option prices is assumed to represent the market sentiment. A first explanation for this argument is that in a risk-neutral world the present price of any contingent claim is equal to the risk-neutral mathematical expectation of its payoff at maturity discounted to the present. The risk-neutral mathematical expectation is calculated with respect to a risk-neutral probability measure, which represents the beliefs of the market. For example if the market is eager to insure against a depreciation of the underlying asset then traders would buy put options, driving their prices upwards. Then put options with low exercise prices would be more expensive than if they represented a fair bet on depreciation. In that case, the risk-neutral probability of a depreciation should be higher than the market's subjective probability.

Since 1978, a lot of progress has been made in the study of implied distributions and researchers have been orientated towards various directions, applying a great variety of promising methodologies.

A very representative classification of implied methodologies can be found in Bahra (1997). Four different approaches are identified: (i) assumptions are made about the stochastic process that governs the movement of the underlying asset and the RND function is derived from it; (ii) the risk neutral density function is estimated non-parametrically, without any restrictions on either the price dynamics, the option pricing function or the terminal distribution itself; (iii) semi-parametric methods, that include the expansion methods and methods where a parameterisation is made either directly to the option pricing function itself or to the implied volatility smile curve;

(iv) a parametric assumption is made for the risk neutral distribution and the parameters are recovered through a minimisation of the distance of the observed option prices and prices postulated by the assumed parametrical form of the distribution

All of the above approaches are based to a great extent on equation (20) developed by Breeden and Litzenberger. Due to this common starting point they share a common characteristic. They are all sensitive to data limitations since a great number of available strikes are needed if we are to obtain a valid approximation of the risk-neutral density of the underlying asset. Furthermore, at-the money options are usually the most actively traded contracts. Out-of-the money contracts are more thinly traded which implies that arbitrage forces which act to preclude arbitrage opportunities from arising, may not be so effective for these contracts. In addition bid-ask spreads are usually a large percent of the premium and thus distort the underlying economic price.

Furthermore, out-of-the money options contain information about the tails of the distribution. This has as a consequence that the fitting of the implied distributions at this regions "...becomes more art than science" as Melick (1999) notices. Often inference in these regions is greatly dependent on the estimation technique used and naturally on the distributional assumption made. Due to the specific lack of data one has to rely on extrapolation techniques rather than on interpolation ones, in order to complete the required spectrum of strike prices. Extrapolation techniques, though, tend to be sensitive to the algorithm used to apply them.

3.1 Stochastic process assumptions

Bates (1991) assumes a jump diffusion model with a Poisson asymmetric jump component to describe the dynamics of S&P 500 futures options, in a study examining whether the crash of '87 could have been predicted using option price implied information. He applied similar methodologies also in 1988 and 1995 in two other studies using the same dataset. Malz (1995) also assumes a jump diffusion model but with this time with a Bernoulli distributed jump component, to estimate realignment probabilities in the ERM mechanism in study with over the counter data for the sterling/mark options market.

Assuming a specific form for the stochastic process that governs the movement of the underlying asset has as an advantage that one can examine dynamic hedging strategies and possibly test their performance. This is not possible if an assumption is made about the terminal risk neutral distribution due to the static nature of such an assumption. Furthermore, since a stochastic process is consistent with a unique terminal distribution, if the process is adequately well behaved then it is possible to estimate this ending distribution.

3.2 Non-parametric methods

The Breeden and Litzenberger formula is very general as no assumptions are made as to which is the stochastic process that the underlying asset follows. The only assumption that needs to be made is, aside from the perfect markets assumption, that the call option price function be twice differentiable. Even this assumption, however, can be relaxed, as it will be shown in the next paragraph in a discrete time environment. No assumptions need to be made about the investors' preferences or beliefs.

The major difficulty that we face, when it comes to actually implementing this approach, is that there is a shortage of available options in the market. If the call pricing function is not known then the second derivative in equation (6) is estimated by a second order difference approximation. Consider a long butterfly spread with three option with strikes X, $X\text{-}\Delta X$, $X\text{+}\Delta X$, where ΔX is a small quantity. Then the price of a long butterfly is equal to:

$$C_b=(C(X+\Delta X,t)-C(X,t))-(C(X,t)-C(X-\Delta X,t)) \quad \textbf{(21)}$$

Taking appropriate weights on the three options equation (21) becomes:

$$C_b=\frac{(C(X+\Delta X,t)-C(X,t))-(C(X,t)-C(X-\Delta X,t))}{\Delta X^2}\approx q(X) \quad \textbf{(22)}$$

Taking the limit as ΔX becomes infinitesimally small expression (22) approximates the second derivative of equation (20).

It is obvious that in order to implement equation (22) there should be a continuous spectrum of strike prices available to traders. If that were the case they would be able to extract the probability distribution from (20) just by replacing observed option prices into the second derivative of equation (22). When available strike prices are far apart then the second order difference is no longer a good approximation of the second derivative of the call pricing function with respect to the strike price.

If a sufficient range of exercise prices was available then the state price density q could be directly estimated by discretising formula (22). Ait Sahalia & Lo

(1995), proposed a non-parametric approach and apply the Nadaraya-Watson kernel estimator to estimate the entire call option pricing function instead. This estimator under certain regularity conditions can be shown to converge to the call function for large samples. It is also free of the typical joint hypotheses and asset price dynamics and risk premia that are typical of parametric arbitrage models, or on preferences in the equilibrium approach in derivatives pricing. However, the convergence is slow because of the curse of dimensionality, i.e. the large number of parameters in the function for the call price and because of other additional factors. It is a particularly data intensive method which makes it practically unimplementable when there is only a fairly limited amount of strike prices available in the market.

Two other papers by Buchen & Kelly (1996) and Stutzer (1996), have proposed a non-parametric way to estimate the state price density based on the maximum entropy method. The entropy of a probability density is defined as:

$$S(p) = -\int_0^\infty p(x)\, ln(x)\, dx$$

the maximum entropy method chooses among all the possible densities the one that maximises the quantity *S(p)* and satisfies at the same time the constraint:

$$\int_0^\infty p(x)\, dx = 1$$

In addition to that, the density is required to reproduce correctly the observed option prices. For example if the recovered density is named *q(S)* then for call options C_i it must be:

$$C_i = e^{-r(T-t)} \int_0^{\infty} \max(S - K_i, 0) q(S) dS$$

However, the absence of smoothness constraints has as a drawback that the recovered density is usually "bumpy", which is contrast with the high degree of smoothness required in kernel regression techniques.

A slight variation of the Breeden and Litzenberger method was proposed by Neuhaus (1995), who used option prices to estimate the cumulative distribution function of the underlying asset. Differentiating once the call pricing function with respect to the strike price we get:

$$C = e^{-r(T-t)} \int_0^{\infty} max(S_T - K, 0) q(S_T) dS_T \Rightarrow$$

$$C_K = -e^{-r(T-t)} \int_K^{\infty} q(S_T) dS_T$$

which can also be written as:

$$-e^{r(T-t)} C_K = p(S_T \geq K)$$

A probability at a point is derived by calculating an integral that contains the point at which we are interested in calculating the probability, for example:

$$p(X) = p(X - 0.5\delta \leq S_T \leq X + 0.5\delta) = -e^{-r(T-t)} \int_{X-0.5\delta}^{X+0.5\delta} q(S_T) dS_T$$

where δ is a small quantity.

Using this approach he derives implied distributions from LIFFE options.

Nakamura and Shiratsuka (1999) conduct a similar study for Japanese options' markets examining three volatile periods.

Andersen and Ratcliffe (1997) in a deterministic volatility model, discretise the deterministic volatility grid onto a pre-specified rectangular grid in time and asset prices. They use finite difference methods to price the options and highlight the differences in using different choices for the finite difference scheme, such as implicit versus semi-implicit. They also point out the computational advantage of deterministic volatility models over implied trees when they are applied for the pricing of exotic options with discontinuous payoffs, such as barrier or digital options. Bodurtha and Jermakyan (1996) and Lagnado and Osher (1997) both within a deterministic volatility model, require that the volatility surface be smooth. However, these methods are rather computationally expensive as both sets of authors recognise.

3.3 Semi-parametric methods

In contrast to non-parametric methods, semi-parametric methods place some structure either on the distribution or on the option pricing function itself or alternatively on implied volatility.

Shimko (1993) proposed an approach where implied volatility was assumed to be a function of the strike price. The relation between these two variables is evident if one plots implied volatility against strike price. The resulting graph usually has a U-shape suggesting that implied volatility could be a quadratic function of the strike price. Following this observation Shimko specified parametrically the volatility curve assuming that it is a parabolic function of the strike price.

$$V=A_0+A_1X+A_2X^2 \tag{23}$$

After recovering the parameters for the parabolic expression (23), he used the volatilities calculated from the formula as inputs to the B&S formula. He was then in a position to differentiate the call pricing function with respect to the strike price and recover the RND function. He defined the tails of the distribution where there were no strikes available to be these of the lognormal distribution so as to have the complete shape of the probability distribution. Shimko's method was further enhanced by Malz (1997) in a study using over the counter risk reversal dollar mark data. He fitted a quadratic polynomial to the implied volatilities but across deltas instead of across strike prices as Shimko did. By following this approach he avoided having to assume that the tails of the distribution coincide with these of a lognormal. Brown and Toft (1999) further extended Shimko's approach by using seventh order splines.

Campa Chang and Reider (1998) use cubic splines to fit the volatility smile. Aparicio and Hodges (1998) use cubic B-splines that belong to a particular family of polynomials used for the segments. Rosenberg and Engle (1997) use a polynomial fitted to the log of the smile, which then automatically prevents negative implied volatilities. Rosenberg (1996) extends this method to the bivariate case. Jackwerth (1999) maximises the smoothness of the smile and can then control the trade-off between option price fit and smoothness explicitly.

Alternatively, researchers, assumed parametric forms for the call pricing function itself. Jarrow and Rudd (1982) proposed a model where the call price was equal to the Black & Scholes price plus two correction terms that adjusted the call price for observed excess skewness and excess kurtosis compared to that of the lognormal distribution. They applied an Edgeworth series expansion around the lognormal distribution to infer the state price density. No assumption needed to be made about the actual functional form of the distribution. Instead only the first four moments of the distribution are estimated empirically and then the option price can be estimated using this additional information.

The Edgeworth series expansion method belongs to the family of expansion models. The general methodology can be stated as follows. The state price density is considered a general probability distribution:

$$P(S_T - S_0 \leq x) = P_0(x) + \sum_{k=1}^{\infty} u_k P_k(x)$$

The first term in the above expression is the approximating distribution around which the true risk neutral distribution is assumed to lie. Typically this is a lognormal distribution. The expansion is then truncated at a finite order and the remaining expression gives a parametric form for the state price density. If this distribution is analytically tractable then closed form solutions for the call and put option prices can readily be obtained.

In the Edgeworth series expansion case if the known approximating distribution is denoted by $p(F_t)$ and the true distribution by $g(F_t)$, then the latter can be expressed in terms of $p(F_t)$ as:

$$\begin{aligned} g(F_t) = p(F_t) &+ \frac{(\kappa_2(g) - \kappa_2(p))}{2!}\frac{d^2 p(F_t)}{dF_t^{\,2}} - \frac{(\kappa_3(g) - \kappa_3(p))}{3!}\frac{d^3 p(F_t)}{dF_t^{\,3}} \\ &+ \frac{((\kappa_4(g) - \kappa_4(p)) + 3(\kappa_2(g) - \kappa_2(p))^2)}{4!}\frac{d^4 p(F_t)}{dF_t^{\,4}} \end{aligned} \quad \textbf{(24)}$$

where

$$\kappa_1(g) = \kappa_1(p)$$

In expression (24) the terms $\kappa_i(g)$ and $\kappa_i(p)$ are the ith order cumulants of the true and the approximating distribution respectively. Cumulants are similar to moments[9]. In fact, the first cumulant of a distribution is equal to the mean, and the second is equal to the variance. The third and fourth terms in the above expansion adjust the lognormal distribution $p(F_t)$ for differences in skewness and kurtosis from the observed distribution. Jarrow and Rudd estimated statistically the moments of the true distribution. Corrado and Su (1997) estimated these parameters implicitly, an approach also followed by Jondeau and Rockinger (1999). Flamouris and Giamouridis (2000) further enhanced the method by extending the model to take into account the early exercise feature of American options.

The way to express this approximating distribution, however, is somewhat arbitrary. Even in the case when this distribution is the simple lognormal there are many choices as to how to define which exactly from the family of lognormals should the distribution be. This means that the first and second moments that define a lognormal distribution are up to the investor to decide how to choose them. Furthermore it is an approximate solution but it does not provide any a priori bounds on the degree of approximation.

A different expansion method has been used by Backus, Foresi and Li (1997) and Jondeau and Rockinger (1998), namely that of the Gram-Charlier expansions. A Gram-Charlier expansion, similarly to an Edgeworth series expansion, generates an

[9] The characteristic function of a probability distribution $\phi(t) = E(e^{itx})$ is calculated with the following Taylor series expansion $\phi(t) = \exp\left\{\sum_{r=1}^{\infty} \kappa_r t^r / r!\right\} = \sum_{r=0}^{\infty} \mu'_r t^r / r!$
Whereas μ_j (moments) are the coefficients of $(it)j/j!$, κ_j are the coefficients of $(it)j/j!$ in $\log\varphi(t)$.

approximate density function for a standardised random variable that differs from the standard normal in having potentially nonzero skewness and excess kurtosis.

A Gram-Charlier expansion defines an approximate density for S_T by:

$$g(S_T) = p(S_T) - \gamma_{1n} \frac{1}{3!} \frac{d^3 p(S_T)}{dS_T{}^3} + \gamma_{2n} \frac{1}{4!} \frac{d^4 p(S_T)}{dS_T{}^4}$$

which is very similar to the Edgeworth series expansion formula. This expansion method has been found to work well in cases where there is high jump intensity or substantial skewness. Similarly with the Edgeworth series expansion it arises from a truncated Taylor series approximation and many times a short Taylor series can prove to be a poor approximation. We would thus expect that at some situations a Gram-Charlier expansion would price poorly option prices. However, the model has the ability to approximate estimates of kurtosis from jump diffusions when plausible parameter values are used.

Potters, Cont and Bouchaud (1998) have applied a cumulant expansion method aiming to reproduce correctly the volatility smile rather than the option prices themselves. The motivation for this approach was that traders tend to work more with implied volatilities instead of option prices since they are considered to be more stable in time.

Abadir and Rockinger (1997) use confluent hypergeometric (or Keummer's) functions as a basis for the risk-neutral density and derive the option price function across strike prices in closed form. Brenner and Eom (1997) use Laguerre polynomials, which are corrections to a gamma distribution, in the same way the Edgeworth series expansion is a correction to a lognormal distribution.

Finally, the last expansion method that has been used so far for the pricing of options is the method of Hermite polynomials that was introduced by Madan and Milne in (1994). Hermite polynomials form a basis of a Hilbert space, which implies that every function in that space can be written as a linear combination of the basis polynomials. Both the payoff function and the state price density are expressed as a combination of the basis polynomials. The coefficients for the payoff function are readily available for simple call and put options but no expression exists for the density function since it is an unknown function. As Madan and Milne point out: " ...pricing in terms of a Hilbert basis is analogous to the use of discount bonds as a basis for pricing of fixed income securities, or the construction of the branches of a binomial tree in pricing options". Abken, Madan and Ramamurtie (1996) generalised Madan and Milne's work by deriving restriction on the coefficients weighting the Hermite polynomial basis elements that allows us to use all traded option maturities jointly in estimation.

Hermite polynomials being truncated series expansions similarly to the Edgeworth series and the Gram-Charlier expansions, they suffer from the same drawbacks. Furthermore, for some parameter values the distribution function for some of the expansion models tends to take negative values, giving rise to densities with no mathematical or physical meaning. Therefore, constraints should be imposed to the parameters recovered to prevent situations like the ones just described to arise. Jondeau and Rockinger (1998) estimate Gram-Charlier expansions imposing positivity constraints. Flamouris and Giamouridis (2000) estimate Edgeworth series expansions under positivity constraints and in addition under the constraint that the integral of the distribution recovered be equal to one. These two conditions have to be imposed since the distribution recovered from observed option prices is a truncated series, which implies that there is no guarantee that the integral of this approximating distribution function will always be equal to unity or that this function will always be positive.

3.4 Parametric methods

Parametric methods assume a specific form for the risk neutral density and calculate its parameters through an optimisation procedure that usually involves the minimisation of the distance between observed and theoretical option prices. The most commonly parametric assumption made is that the risk neutral density belongs to the family of the k-component mixture of lognormal distributions, first introduced in the options' pricing literature by Ritchey in (1990). Different parameterisations for the risk neutral density function have also been suggested but did not gain the "popularity" of the previously mentioned method.

Theoretical option prices can be obtained using the risk neutral valuation approach. These prices will typically involve the parameters of the assumed distribution. To recover these parameters an optimisation procedure is employed, the most commonly applied one being the minimisation of the sum of squared deviations of the theoretical prices from the observed option prices in the market. Other minimisation criteria have been proposed by Jackwerth and Rubinstein (1996), such as the goodness of fit function, the maximum entropy function and the absolute difference function.

Parametric methods impose a significant amount of structure to the risk neutral density, which is indeed confined to possess the properties of the assumed family of distributions. Thus, it is important to choose a distribution that is flexible enough to be able to capture the diverge characteristics that implied distributions are shown to exhibit, namely the degree of excess skewness and excess kurtosis over that of a lognormal distribution present in the markets. Furthermore, the modelling of the tails of the distribution is greatly dependent again on the type of the distribution we assuming. Parametric models overcome the shortcoming of having to employ extrapolation techniques for the out-of-the money regions of the data at the cost that the form of the distribution will determine to a large extend the behaviour of the distribution at these regions.

This approach could present a serious shortcoming. The evaluation of an option price for an arbitrary underlying distribution may be possible only by numerical integration techniques because of the possible analytical intractability of the particular distribution function. Empirically it may be easy to estimate the moments and other parameters of the distribution, but using that information to compute option prices may be considerably more complicated. This shortcoming can be avoided if a distribution is chosen that is flexible enough to incorporate a wide variety of different shapes and numerically tractable so that it can provide closed form pricing formulas. The mixture of two lognormal distributions demonstrates these two desirable features as it will be shown later in the text, which explains why it has been used extensively as a tool for the recovery of implied distributions.

Another approach that attempts to parameterise the risk-neutral density is the work of Aparicio and Hodges (1998), who use generalised beta functions of the second kind. This family of four parameter distributions was first explored by Bookstabe and McDonald (1987) and includes the lognormal, Gamma and exponential distributions several Burr type distributions and as well as other distributions. Posner and Milevsky (1998) use the Johnson family of distributions to infer the risk-neutral density from option prices. They were first proposed by Johnson, (1949) who translated the normal distribution through logarithmic and hyperbolic sine transformations, while adding two additional parameters. Finally, Sherrick Garcia and Tirupattur (1995), apply the Burr III distribution on a data set on soybean futures. Sherrick, Irwin and Forster (1992) and (1996) use the Burr XII distribution of options on S&P 500 futures.

3.5 Implied Models' Tests

As the "popularity" of implied models is rapidly increasing researchers started building appropriate tests to examine the usefulness of this class of models in terms of practical applicability. Do these models produce hedging strategies that outperform naïve trading rules or the deficient Black & Scholes model?

Perhaps the most commonly referred to paper is that of Dumas Fleming as Whaley (1999) criticising deterministic volatility function models[10]. The authors fit the volatility surface to the observed option prices every Wednesday for a period of five years, and use this surface to forecast option prices one week ahead. Then the following week they observe option prices in the markets and form a mean squared prediction error. The average root mean squared prediction error ranges from 47 to 59 cents for the options with time to maturity of more than seventy days. The authors compare this with a typical spread for S&P 500 options, of about 47 cents. Even though, the performance of this type of implied methodologies does not seem to be

[10] I am referring to this paper also in the discrete time models section

very impressive competing fully parametric models do not necessarily perform much better.

Bakshi, Chao and Chen (1997) fit parametric jump and stochastic volatility models again on the same set of data for a bigger time span. They predict one-day ahead option prices and report average pricing errors of about 40 cents even for their best performing models, when used to predict option prices of contracts with 60 up to 180 days of expiration. Also Jackwerth and Rubinstein (1998) compare implied binomial trees, parametric models (among them a stochastic volatility model) and naïve trader rules of thumb. These naïve trader rules assume that traders simply apply the volatility smile recovered in one day to the option prices of the next day, assuming thus that the shape of the volatility smile remains constant over an one day period. They use again options on the S&P 500 index for options with 135 to 215 days to expiration, and try to predict option prices two weeks ahead. The errors according to their estimates range from 75 cents for the binomial tree to 57 cents for the naïve trader rule and to about 78 cents for the parametric models. However, as it is stated in their paper given that the empirical standard deviation of these errors is 80 to 130 cents, they can conclude that none of these models performs extraordinarily well.

The evidence presented in these two papers shows that there is not a way to distinguish between the different types of models that were being tested. Is this really the case or researchers have been testing the wrong hypothesis?

Both these papers and others that have undertaken the task to examine the usefulness of implied methodologies[11] examine their forecasting performance by comparing option prices as predicted by the model at a point in time with option prices at a later point. According to the author's opinion this is not the proper way to test whether the extended use of implied methodologies in option pricing is justified. Implied methodologies should be used for two main reasons. First, they can accurately recover the market sentiment embedded in option prices. As it was previously mentioned implied distributions are forward looking, thus, they can reflect immediately any impact that a significant external macroeconomic event might cause to option prices. Second, implied models are built in such a way so as to be consistent with observed option prices therefore they can be used for the pricing of exotic derivatives or other over the counter (OTC) products. Pricing these instruments with an implied model we can obtain prices that do not suffer from model inconsistency.

Implied distributions cannot be used for the prediction of option prices since they are static in nature. No assumptions are made for the dynamics that govern the movement of the underlying asset. They can only incorporate the market sentiment for a specific point in time, the option expiration. The fact that researchers find that the option prices as predicted by theses models differ from actual ones by a certain percentage this could possibly that market sentiment changed over the period that elapsed between time zero when the prediction was made and the time point that option prices were observed again. This could be due to the arrival of new information in the market as option prices are forward looking as previously stated, or it could even be due to the fact that traders had a wrong view of what was to come in the first place.

[11] See for example Gemmil and Saflekos (2000)

Implied methodologies should be tested for their ability to recover the market sentiment since this is one of their two main applications[12]. The changes of the shape of the risk neutral density along with a time series of a variety of summary statistics provide the ground for most of the tests that have been made on this direction. However, as Jackwerth (1999) notices, "...none of the researchers provide for a statistical test of the significance or controls for exogenous factors that could cause the change in shape. For example increased volatility could have been driven by a market-wide rise in volatility rather by the particular event of an economic announcement."

The second use of implied models is the pricing of more exotic products consistently with their simple vanilla counterparts. Implied models assume that observed options are priced correctly, not looking for arbitrage opportunities, and proceed for the valuation of the more complicated products. Even if the prediction performance of implied models is not the best it could be for vanilla options as it results from the papers mentioned before this is not what researchers should be looking at. Achieving consistent pricing of exotic with vanilla options implies that when the simple call and put options are used to hedge the more complicated or OTC structure then even if the products included in the hedge may not be correctly priced, the hedge, however, should remain valid and possibly unaffected. This should happen because the exotic and the vanilla instrument are priced using the same model and set of information and they should be expected to deviate from the "correct" prices by a similar amount.

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[12] Flamouris and Giamouridis (2000) contact such a study for an Edgeworth Series Expansion method extracting information from American options.

Alternative tests concerning the applicability of implied models should test for the performance of hedging strategies for exotic options priced from an implied model and hedged with plain vanilla securities. The results of such a test would truly validate or abolish the usefulness of implied distributions.

4 CHAPTER FOUR: RESEARCH OUTLINE AND JUSTIFICATION

In my Ph.D., I will exploit both uses of implied distributions. In Chapter 5 I will apply them for the purpose of extracting information concerning major macroeconomic events. More specifically I will study whether someone could have used information embedded in option prices prior to the sterling pound's exit from the Exchange Rate Mechanism in September 1992, in order to predict this very event.

Consequently, in Chapters 6 and 7, I will exploit the second, and possibly the most important use of implied methodologies, by utilising option recovered information in order to price exotic options in a market consistent way. I will apply an implied model to the pricing of Geometric and Arithmetic Average options and also for the pricing of Basket options.

Throughout my research I am using a methodology that assumes a specific parametric form for the risk–neutral density function and recover its parameters through an optimisation procedure. Among the possible distributional assumptions I chose the family of mixture of lognormal distributions. A variety of reasons suggest that this distribution is a well-suited candidate for both uses of implied distributions.

Implied option pricing literature is growing fast since researches extend and expand the approaches mentioned earlier in the text. This fact urged the need for the development of some kind of testing procedure in order to distinguish the "best" methodology among the existing ones. Implied distributions recovered according to different models were compared in a number of papers. The criteria for the goodness of a distribution more often were the fit it provided to the observed option prices and less frequently its ability to forecast the statistical properties of future data. In most of these papers the mixture of lognormal distributions proves among the best models one could employ to extract information embedded in option prices.

McManus (1999) compares six different methodologies for the recovery of risk neutral densities, a Black & Scholes model, a mixture of lognormal distributions, a Hermite polynomial of order four and a Hermite polynomial of order six, as well as a maximum entropy method. He finds that the mixture of lognormal distributions outperforms the rest of the models using as a comparison metric the models' ability to fit observed option prices. Jondeau and Rockinger (1999) compare the benchmark lognormal model with a lognormal mixtures model, a Hermite approximation of order four, an Edgeworth series expansion, a jump-diffusion model and a stochastic volatility model. They find that the mixture of lognormals model performs better than the rest of the models for short maturity options while the jump-diffusion model performs better for long maturity options. The comparison method they use is the model's ability to capture the excess degree of skewness present in the foreign exchange dataset they are using. Jackwerth (1999) when comparing the mixtures family of models with other parametric methodologies, while stressing the danger of over parameterisation of the data he concludes that "...Still mixtures tend to be somewhat more flexible and are capable of generating a wider variety of shapes for the probability distribution than the generalised distributions...".

Cooper (1999) compares the mixtures model with a smile fitting technique and he finds that the smile technique does seem to outperform in most cases the mixtures model but the results are not clear-cut as to which one is indeed better. The paper tests for the accuracy and the stability of the estimated summary statistics from the risk-neutral densities.

The last paper does not favour the mixtures distribution, however, it does show that it is not always feasible to distinguish in a decisive fashion between competing implied models. Campa, Chang and Reider (1997) compared binomial tree, smoothed volatility smile and mixtures of lognormals methods. Comparing various moments of the implied distributions they concluded that all methods produced similar results. Giamouridis and Tamvakis (2000) report the results of an experiment where a number of researchers have used a common data set in order to estimate summary statistics from recovered implied distributions each with a different methodology. The percentiles as well as the mean, variance, skewness and kurtosis as estimated from the different implied models, did not greatly vary across models. These findings further reinforce the argument that most of the implied methodologies have the ability to reflect market consensus at a satisfactory level.

Further to the results presented previously supporting the mixtures distribution as a valid choice for a base for an implied methodology, this family of distributions has additional merits. The k-component lognormal mixtures models belong to a mathematically tractable class of statistical density functions, making them an appealing candidate for a terminal density function for any underlying asset's process. Closed form solutions for the values of vanilla call and put options do exist in the mixtures framework with the formulae being simply a weighted average of corresponding Black & Scholes prices. The closeness of the mixtures distributions to the Black & Scholes prices is often referred to as the greatest advantage of the method. The k-component lognormal mixtures model is a generalisation of the B&S

model (one could say that it belongs in the B&S "neighbourhood") without losing the analytical tractability of the latter. Ritchey (1986) found this model to be descriptive of the majority of common stock returns.

Finally, it appears as a regular empirical finding in altogether different methodologies that aim to recover implied probability distributions from option prices that bimodal or even trimodal patterns appear in the graphs of the extracted distributions. These patterns are typical of the two- or three- component lognormal class of distributions.

Rubinstein (1994) uses a dataset of S&P 500 index options and recovers probability distributions that exhibit a bimodal pattern with a small hump to the left of the distribution. Rubinstein explains "...the bimodality coming from the lower tail ("crash-o-phobia") is quite common during the post-crash period...".

Bates (1991) employs a jump-diffusion framework and recovers implied distributions, which exhibit multi-modal patterns. A representative graph is presented below:

The Crash of '87: Was It Expected? 1037

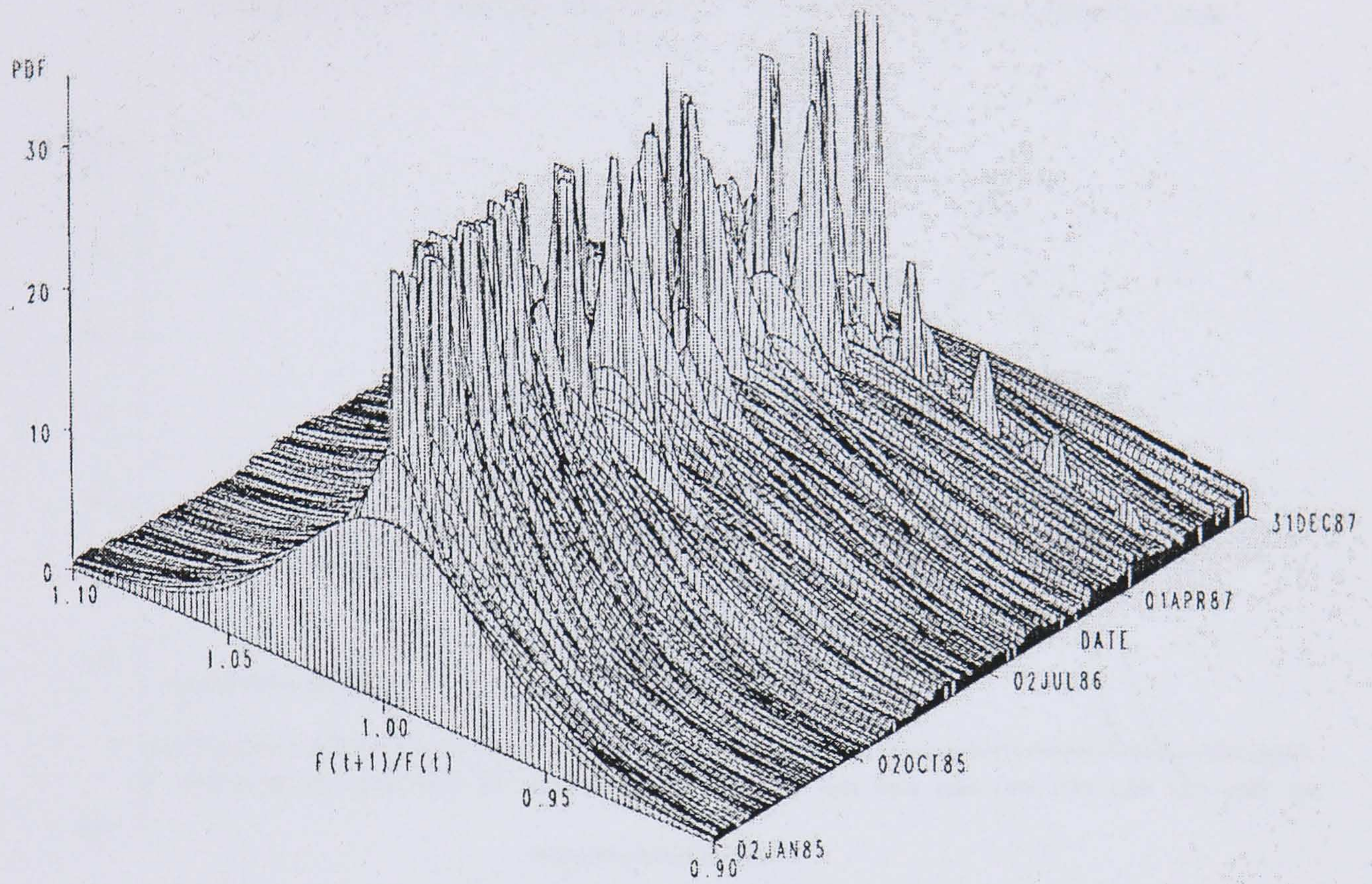


Figure 10. Probability density function of $F_{t+1\ \text{month}}/F_t$ implicit in call and put options on S&P 500 futures, 1985-1987. See notes to Figure 8.

Figure 5: Bates (1991) graph, page 1037.

Jackwerth and Rubisntein (1995) recover implied probability functions from contemporaneous security prices in a non-parametric way. When they plot the distribution recovered by their methodology the resulting graph has the following shape:

Figure 8
`Clamping Down' on the Solution Set for the Closed-Form Smoothness
date: March 16, 1990

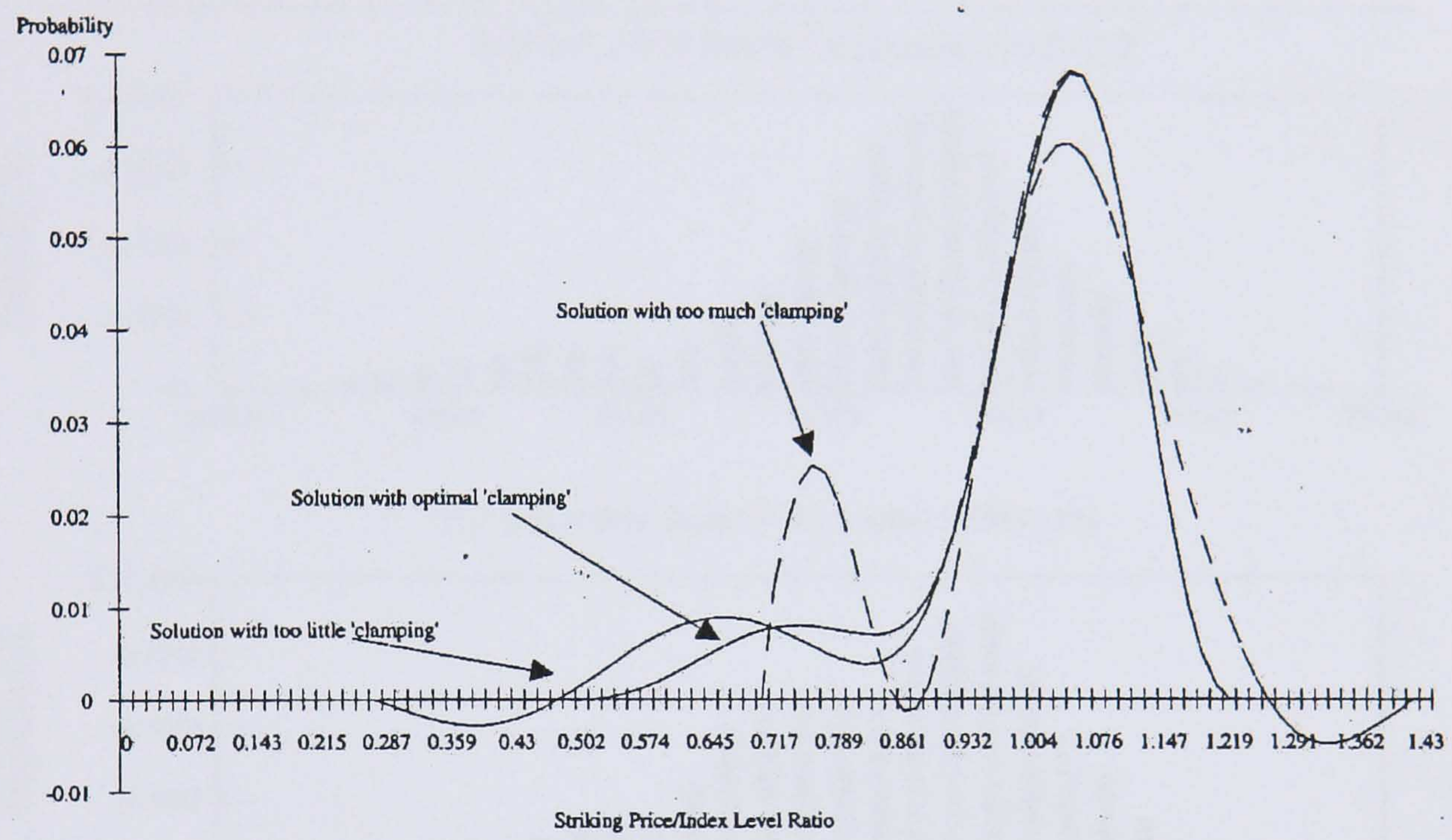
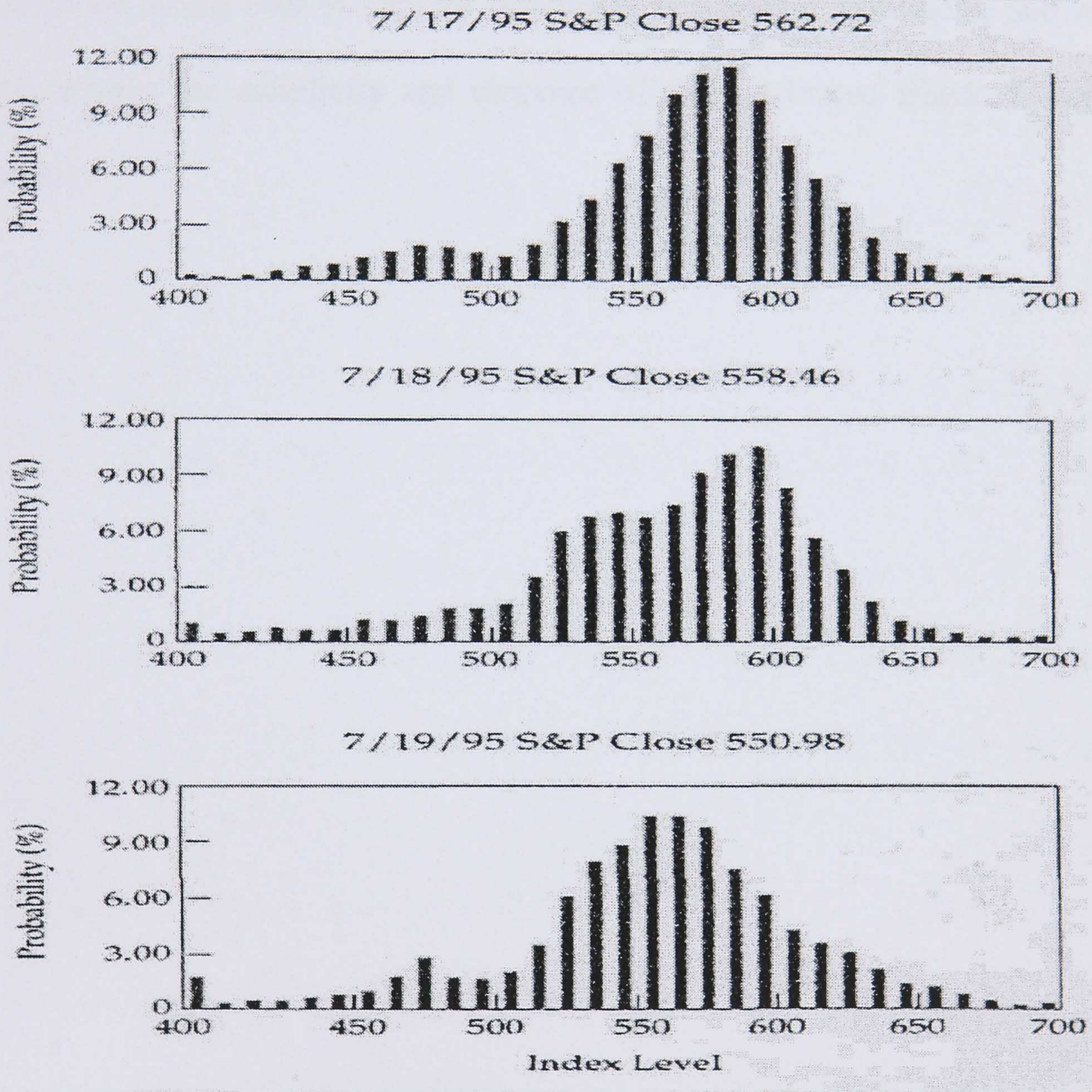


Figure 6:Jackwerth and Rubinstein (1995)

The three lines in the graph depict the plot of the RND with an application of the smoothing criteria they test in their paper. The bimodal pattern is again clearly present in the above plot.

Derman, Kani and Zhou (1996), in a discrete time setting, recover the implied volatility surface and infer risk neutral densities from observed option prices. The recovered discrete distributions greatly resemble distributions belonging to the mixtures of lognormals family.

Figure 9. Implied S&P 500 Distributions on December 31, 1995, based on S&P 500 Implied Volatilities on July 17, 18, and 19, 1995



Note: Assumptions—growth rate = 6 percent, dividend yield = 2.5 percent to year-end.

Furthermore, a jump-diffusion model with Bernoulli distributed jumps is consistent with the k-component lognormal mixtures terminal risk neutral density. A terminal distribution though is not consistent with a unique stochastic process while the reverse is true. A stochastic process that governs the dynamics of the movement of the underlying asset is consistent with a unique terminal risk neutral distribution. In Chapters II and III, **Part 2**, a Bernoulli jump diffusion stochastic process will be utilised for the evaluation of exotic derivatives.

The compelling evidence presented above supporting the choice of a mixtures of lognormal distributions, directed me towards selecting this specific distributional family for the rest of my research. It offers the flexibility required from a parametric form assumption being able to capture accurately the market sentiment, and at the same time, retains the simplicity and elegance of the celebrated Black & Scholes (1973) model.

5 CHAPTER FIVE: USING IMPLIED PDFs FOR THE PREDICTION OF MACROECONOMIC CRISES

Implied methodologies have numerous applications, the most popular being the direct recovery of the market sentiment, as it has already been argued. Investors make better-informed judgements when they compare their own views of future underlying asset movements with the market consensus. Policy makers, on the other hand should find implied modelling most appealing, since they can have a strong understanding about market's views, and consequently base their decision making upon this information. In the foreign exchange market, central banks are very concerned with future exchange rate movements since there might be a possibility of an intervention in order to maintain stability, for the domestic currency[13].

Implied models are, therefore, an extremely useful tool for policy makers, since market beliefs on future moves of the exchange rates can be reflected in current option prices. Furthermore, the use of implied methodologies is more advantageous over the alternative of simply using the futures markets to extract information concerning subsequent dates. Futures contracts merely reflect the market expectation of the future exchange rate, which corresponds to the mean of the risk neutral distribution of the exchange rate, not revealing any information on the variance or higher moments. These higher moments, though, contain information concerning the level of dispersion providing an indication of the uncertainty present in the market

[13] To the author's knowledge implied methodology techniques are used extensively in the Bank of England as well as in the Bank of Italy.

place. Thus, information embedded in option prices can be used as a guideline for exchange rate policy.

Malz (1997) conducted work in the exchange rate area where he used OTC foreign exchange options, quoted in delta terms, to extract the probability distribution of the underlying exchange rate. His method serves as an improvement to that of Shimko (1993), since the parabolic expression of equation (23) was replaced by a more complex function and the simplification assumption that the tails of the distribution are approximated with that of the lognormal was not made.

Also Campa, Chang and Refalo (1999) for the BIS (Bank of International Settlements) Workshop employed non-parametric ways to extract implied expectations of emerging markets exchange rates.

For the first chapter of my thesis, I am going to study the hypothesis of whether such a technique could have provided useful information for the prediction of the macroeconomic event of the sterling pound's exit of the ERM. Many researchers have tried to examine whether other events could have been predicted using information coming from the options markets. Most of the work in this area is concentrated on the 1987 crash[14], a major event in recent financial history.

Since its inception in 1979 the European Monetary System (EMS) has provided an interesting example of a formal Exchange Rate Mechanism (ERM) and a framework for international policy co-ordination. Intra-European exchange rates are allowed to fluctuate only within official bilateral limits, and this commits national monetary authorities to maintain exchange rates within these limits through exchange rate market interventions. The ERM is an exchange rate target zone with narrow bands. Until 1987 realignments were frequent in the EMS, but from 1987 until 1992 there were no EMS realignments[15]. However in the autumn of 1992 and during the EMS crisis two currencies, UK sterling and the Italian lira, were suspended from the ERM.

Devaluation fears and realignment probabilities have been studied by Svensson (1993) and Rose and Svensson (1994) using statistical techniques while Campa and Chang (1996), Campa, Chang and Reider (1997), Malz (1996) used information embedded in option prices, to asses the target-zone credibility or to estimate realignment probabilities.

In my study an assumption will be made about the parametric form of the terminal risk neutral density function and then the parameters will be retrieved from option prices through an optimisation procedure. This assumption is that the distribution of the underlying currency belongs to the family of mixture of lognormal distributions. Making an assumption for the terminal risk neutral distribution rather than for the stochastic process, by which the underlying is driven, provides greater flexibility. The reason for that is that a given stochastic process for the underlying security implies a unique terminal distribution while the converse is not true. That is, a given terminal risk neutral distribution is consistent with a number of different stochastic processes. The drawback of this methodology is that it does not supply any information on the dynamic evolution of the underlying security price. It only

[14] See Bates (1991) for a study using US data (options on S&P 500 futures) and Gemmil (1996) for a UK study using options on FTSE 100 index.

[15] The realignment of the Italian lira, that took place in January 7, 1990, was a technical step to ease entry into the narrow ± 2.25 fluctuation bounds.

provides us with the view of the market at specific moment in time for another specific point in time. However, since the goal is to examine whether the pound's exit from the ERM could have been predicted simply by observing the options' market, the information provided from the model will be adequate.

Further to the characteristics of the mixture of lognormals stated earlier, that make it a very appealing candidate for a parametric model describing the risk-neutral density of the underlying asset, a number of additional reasons establish its dominance especially for the study of foreign exchange options. The lognormal mixtures framework is consistent with Friedman and Vandersteel's evidence (1982) evidence that exchange rates distributions' parameters may be time-varying. Boothe and Glassman (1987) lent support to the idea that even if the distributions' parameters are time-varying the class of distributions which best fits the data is not. They also find in that paper that the mixtures of two lognormals describe best the daily price changes of the UK/US exchange rate. Tucker and Pond (1988) find that the mixture of normal densities performs better than a scaled t-distribution[16] as a candidate process to explain exchange rate changes, and even though they find that a mixed-jump diffusion model would perform ever better, the component lognormal model still performs well.

Ritchey's (1990) call option valuation model for discrete normal mixtures will be employed for the analysis. This model has been used before by Melick and Thomas (1996) in an application to oil futures and by Bahra (1996) in an application to U.K. short-term rates, long bond futures and in LIFFE equity index options. Also

Mizrach (1996) in a study using ERM data for the period 1992, examines, within this framework, whether the British pound and the French frank devaluation could have been foreseen.

5.1 The model

The Black and Scholes GBM assumption implies that the RND function of S_T, $q(S_T)$ is lognormal with parameters a, b i.e.

$$q(S_T) = \frac{1}{S_T b\sqrt{2\pi}} e^{\frac{-[\ln(S_T)-a)]^2}{2b^2}} \tag{25}$$

According to the Geometric Brownian Motion assumption, though, as stated in equation (1), it is:

$$dS_t = \mu S_t dt + \sigma S_t dZ$$

The correspondence of the parameters of equations (1) and (25) is shown below:

$$a = \ln S_0 + (\mu - 0.5\sigma^2)\tau, \qquad b = \sigma\sqrt{\tau}$$

If a k-component lognormal mixtures distribution is considered then the probability density function will be given by the following formula:

$$q(S_T) = \sum_{i=1}^{k} [\theta_i L(a_i, b_i; S_T)] \tag{26}$$

For (26) to be a valid density function it should integrate to one and remain non-negative everywhere. Hence, the probability weights θ_i, should satisfy the conditions:

- $\sum_{i=1}^{k} \theta_i = 1$,

[16] Rogalski and Vinso (1977) conclude that the scaled t-distribution provides a reasonable fit for

(27)

The value of a European call and put option, within the 2-lognormal mixtures model, it can then be easily found, if we invoke the risk-neutral valuation principal, by integrating the payoff function of the derivative contract over the risk-neutral density function:

Specifically, for a call option it is:

$$c(X,\tau)=e^{-r\tau}E_{Q}\left[\max(S_T-X,0)\right]=e^{-r\tau}\int_{X}^{\infty}q(S_T)(S_T-X)dS_T$$

and for a put option

$$p(X,\tau)=e^{-r\tau}E_{Q}\left[\max(X-S,0)\right]=e^{-r\tau}\int_{0}^{X}q(S_T)(X-S_T)dS_T$$

Assuming that the two lognormals that compose the implied RND function $q(S_T)$ are $L(a_1,b_1;S_T)$ and $L(a_2,b_2;S_T)$ and that θ is the weight, the function that expresses the value of a call option is:

$$c(X,\tau)=e^{-r\tau}\left\{\theta\left[\mathrm{e}^{a_1+\frac{1}{2}b_1^2}N(d_1)-XN(d_2)\right]+(1-\theta)\left[\mathrm{e}^{a_2+\frac{1}{2}b_2^2}N(d_3)-XN(d_4)\right]\right\} \qquad \textbf{(28)}$$

where $d_1=\dfrac{-\ln X+a_1+b_1^2}{b_1}$,

$d_2=d_1-b_1$,

---

floating rate periods. Westerfield (1977) finds that the distribution of 6 exchange rates is highly non-normal.

$$d_3 = \frac{-\ln X + a_2 + b_2^2}{b_2},$$

$$d_4 = d_3 - b_2$$

Similarly the function that gives us the value of a put option under the assumption of a 2-component lognormal mixtures distribution is:

$$p(X,\tau) = \mathrm{e}^{-r\tau}\left\{\theta\left[-\mathrm{e}^{a_1+\frac{1}{2}b_1^2}N(-d_1) + XN(-d_2)\right] + (1-\theta)\left[-\mathrm{e}^{a_2+\frac{1}{2}b_2^2}N(-d_3) + XN(-d_4)\right]\right\} \quad \textbf{(29)}$$

with d_1, d_2, d_3, d_4 as before.

In my study I have set *k=2* and *k=3*, studying the cases of a 2-component mixtures distribution and of a 3-component mixtures distribution. If a k-component mixtures distribution is used with *k* greater than 3, then the number of parameters present in the pricing equations, rises significantly. For a four-component mixtures model, for example, the estimation of 11 parameters is required, while for a five-component model the corresponding number rises to 14. Despite the fact that a model with a large number of parameters can describe the data with an increased level of accuracy, the danger that we might over-parametrise the observed option prices is present. The gain in descriptive ability comes at a cost of potential overfitting. This trade off suggests that we should give up some of the explanatory power of the model and favour a more parsimonious representation of the probability density function.

It should be noted here that the results found using the 3-component mixture distribution were almost identical to the ones obtained when the 2-component mixture distribution was used. For this reason I will only refer from now on to the 2-component mixtures distribution. There were 9 parameters to be estimated for the 3-component lognormal mixtures model and only 6 for the 2-component one. The limited amount of data available further restricted me to use the simpler model, a restriction that as it was just mentioned did not prove to be a serious one.

A weighted sum of independent lognormal density functions ensures that the fitted call pricing function is monotonically decreasing and convex in exercise price and is therefore consistent with the absence of arbitrage. Each lognormal density function is completely defined by two parameters. The values of these parameters and the relative weighting applied to the several density functions together determine the overall shape of the mixture implied RND function.

In my study I am dealing with American style options. American options can be exercised at any point in time between now and maturity. This implies that the risk neutral valuation argument cannot be applied in this case since the value of the option does not depend solely on the payoff of the contract at maturity. This introduces an additional degree of complexity, since no closed form solution exists, such as (28) and (29) for this type of options.

Researchers have dealt with the American option valuation problem in a number of ways. Binomial trees are one of the common approaches followed. However, convergence is slow on a binomial tree and faster methods are required for trading purposes. Barone-Adesi and Whaley (1987) developed an approximation method which is the fastest existing method but at the same time the least accurate one. Broadie and Detemple (1996) developed a method based on an approximation of a lower (LBA) or of a lower and an upper bound (LUBA). In that paper they compare, in terms of speed and accuracy, their method with the method of lines from Carr and Faguet (1994), the integral method from Kim (1990), the modified two-point Geske and Johnson (1984) method of Bunch and Johnson (1992a and b) and with various other existing American option valuation methods.

All these methods, though, share a common characteristic. They are all based on the assumption that the underlying asset does follow a Geometric Brownian Motion, or in other words, that the terminal risk-neutral distribution of the underlying asset is a single lognormal distribution. This has as an immediate effect that none of these methods can be applied within the 2-component lognormal mixtures framework. The binomial method is perhaps, the only one that could be adjusted within this framework if the method described in Brown and Toft (1999) was followed and a risk-neutral distribution was imposed on the terminal nodes of the tree. Nevertheless, the algorithm could still be quite slow to be used for practical purposes.

An efficient method that can be applied within the framework of lognormal mixtures distributions was developed by Melick and Thomas (1996)[17]. The distributional assumption they adopted was that of 3-component lognormal mixtures distribution. They introduced bounds for the American option values that depend only on the terminal risk-neutral distribution of the underlying asset. Due to the fact that the bounds they developed depended only on the ending distribution, they can be applied for any choice of the latter[18]. Namely the upper and lower bounds for a call and a put option are given by:

$$C^u(X,t) = \max\{E_t[S_T] - X, e^{-rdt} E_t[\max(0, S_T - X)]\}$$

[17]Melick and Thomas (1996) introduced these bounds for options on futures. The same authors have used the same set of bounds in Melick and Thomas (1998) for American currency options.

[18] Flamouris and Giamouridis (2000) apply these bounds to recover risk neutral probability density functions from American options assuming an Edgeworth series expansion around a lognormal distribution for the terminal distribution of the underlying asset.

$$C^{l}(X,t) = \max\{E_t[S_T] - X, e^{-rt} E_t[\max(0, S_T - X)]\}$$

$$P^{u}(X,t) = \max\{X - E_t[S_T], e^{-rdt} E_t[\max(0, X - S_T)]\} \quad \textbf{(30)}$$

$$P^{l}(X,t) = \max\{X - E_t[S_T], e^{-rt} E_t[\max(0, X - S_T)]\}$$

The first term in the *max* operator in the above bounds is the expected payoff of the option at maturity whereas the second term is the value of an identical European-style option under the assumption that the underlying asset's terminal risk neutral distribution is a two-component lognormal mixtures. In other words, the second term in the *max* operator is the expression in the right-hand-side of equations (28) for a call, or, (29) for a put, without being discounted. The upper and lower bounds essentially differ only in the way the expected payoff is discounted. They are directly related to the time in which uncertainty about the ending value of the underlying asset's price will be resolved; the sooner the uncertainty will be resolved, the more valuable the option.

The value of an American option will be a weighted average of the upper and lower bound. For example for the American Call we would have:

$$C^{amer}(X,t) = wC^{l}(X,t) + (1-w)C^{u}(X,t) \qquad 0 \le w \le 1 \quad \textbf{(31)}$$

A possible drawback of this method for handling the early exercise feature of American options is that the weights in expressions (31) are assumed to be the same across different strike prices and across calls and puts. This implies that the options cannot be priced exactly since, in reality, the weight will not be the same for all these options. Melick and Thomas (1996) introduce an extra weight depending on the moneyness of the options. They use a different weight for out-of and for in-the-money options. However, given the tightness of the bounds and the optimisation that will be used to recover the parameters of the terminal risk neutral density from market option prices, the additional error that will arise from assuming identical weights across different options, will not be by any means significant. The parameterisation in (31) reduces the number of coefficients to be estimated without introducing a significant cost on terms of fitting performance.

In order to solve for the terminal risk neutral density function we have to estimate its defining distributional parameters from market prices of options[19]. In the case of the two-component lognormal mixtures model, there are six parameters overall to be estimated from American option prices observed in the market: five parameters $(a_1, a_2, b_1, b_2, \theta)$ in (28) and (29), and the extra weight parameter (i.e. w) that gives the value of the American option as a linear combination of the upper and lower bound in equations (31). The set of parameters, which causes the values of the theoretically computed American options to be, in the Least Squares sense, closest to the observed market values corresponds to the recovered risk-neutral density function.

There are other distance-criteria that could also be used, such as the goodness of fit function or the maximum entropy function, or the absolute difference function[20]. The least squares function, however, is the most widely used criterion, and it is the criterion that will be used in this study. Since both calls and puts are priced off the same underlying distribution, we should include both sets of prices in the following minimisation problem:

$$f = \min_{a_1, a_2, b_1, b_2, \theta, w} \sum_{i=1}^{n} \left[c(X_i, \tau) - \hat{c}_i\right]^2 + \sum_{i=1}^{m} \left[p(X_i, \tau) - \hat{p}_i\right]^2 \qquad \textbf{(32)}$$

[19] For a study of how to estimate, the parameters of a two-normal mixtures model, from time-series data of the underlying, see Cohen (1967).

[20] See Jackwerth and Rubinstein (1995) for a more detailed description of each criterion.

subject $b_1, b_2 \succ 0$, $0 \leq \theta \leq 0.5$ and $0 \leq w \leq 1$.

over the available strike price range for calls and puts. Due to the symmetry of the functional form of the mixtures distribution and in order to increase the computational speed and accuracy, the θ parameters is constrained to vary only between zero and 0.5.

In equation (32) $\hat{c}_i$ and $\hat{p}_i$ denote the fitted values for a call or a put option that can be obtained for a specific strike price and time to maturity, using a certain set of the six distributional parameters. The terms $c(X_i, \tau)$ and $p(X_i, \tau)$ denote the market values of the calls and puts respectively.

For the optimisation procedure to provide the desired results, the restriction that the mean of the recovered distribution be equal to the risk-neutral mean (the forward rate), is imposed.

When running the minimisation of the objective function f in (32), it is a requirement that all the functions included have an analytic derivative, or alternatively stated, they should be smooth functions. However, in the bounds for the value of the American options there is a max operator present, which is not a smooth function.

To account for that the logistic approximation is being used:

$$\text{logitmax}(x, y) = \frac{1}{1 + e^{-5(x-y)}}$$

The maximum function is then approximated by the following expression:

$$\max\{x, y\} \approx \log it \max[x, y] \cdot x + (1 - \log it \max[x, y]) \cdot y$$

This approximation does have analytic derivatives of any order and it approximates accurately the function that returns the maximum of the variables x and y.

5.2 The Data

The data set consists of daily series of American-style US dollar/British pound (\$/£) options traded on the PHLX exchange, for the month September for the year 1992. The data were provided from the Philadelphia Exchange. The options are for the purchase of one pound with US dollars, where each contract is for £31,250.00. Options are traded with maturity dates in March, June, September and December for up to nine months into the future. There are also maturity dates available for each of the next two consecutive months. Non-liquid option contracts or options that were quoted with the same premium while having different strikes were excluded from the data sample. Closing prices were used[21]. One could argue that closing prices might introduce a non-synchronous trading problem. However, this problem was dealt with in the best possible way since in the data set along with the option price, the spot rate when the transaction was made is also recorded, which enabled the entry of the appropriate exchange rate when for the estimation the parameters of the implied distribution.

Finally, for discounting purposes, the US and UK Eurorates were used as an approximation of the risk-free US and UK rates and interpolation of the two closest to the options expiration rates was carried out to obtain the appropriate rates where necessary.

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[21] For a very detailed discussion on data exclusion see Rubinstein (1985).

5.3 Results

The implied density functions for the October 1992 expiration contracts on different trading dates in September 1992 are shown in Figure 7 to Figure 13 below. It is evident from these figures that the two-lognormal mixtures model can accommodate a wide variety of possible shapes, which are due to variation in its defining distributional parameters caused by changing supply and demand. More specifically this parametric approach is able to incorporate a wide range of possible scenarios, including a situation in which the market has a bimodal view about the terminal distribution of the sterling-pound. A bimodal pattern in the implied densities serves a clear indication that there is a high level of uncertainty among traders.

Figure 7:Implied risk-neutral density for the 4th of September

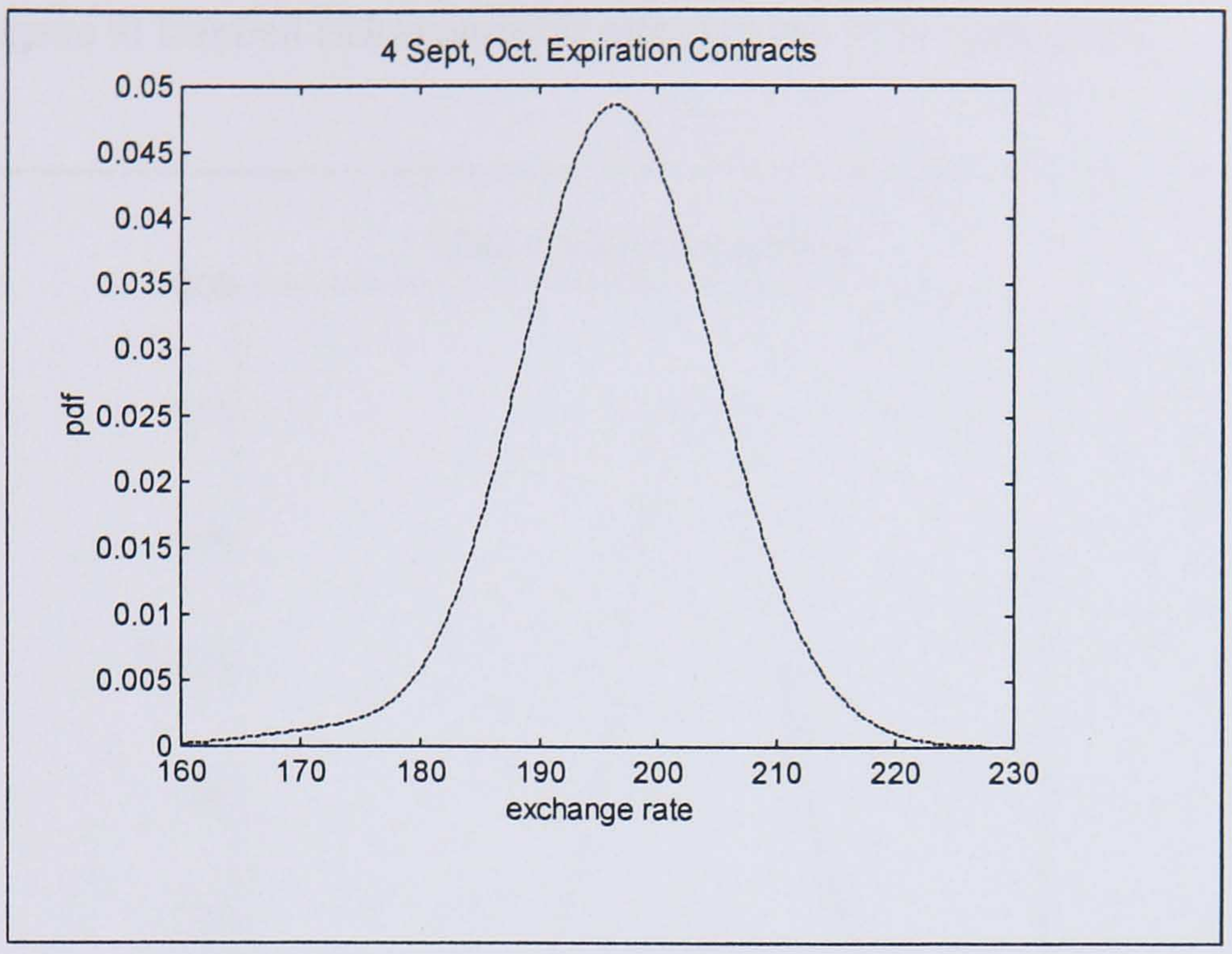


Figure 8: Implied risk-neutral density for the 8th of September

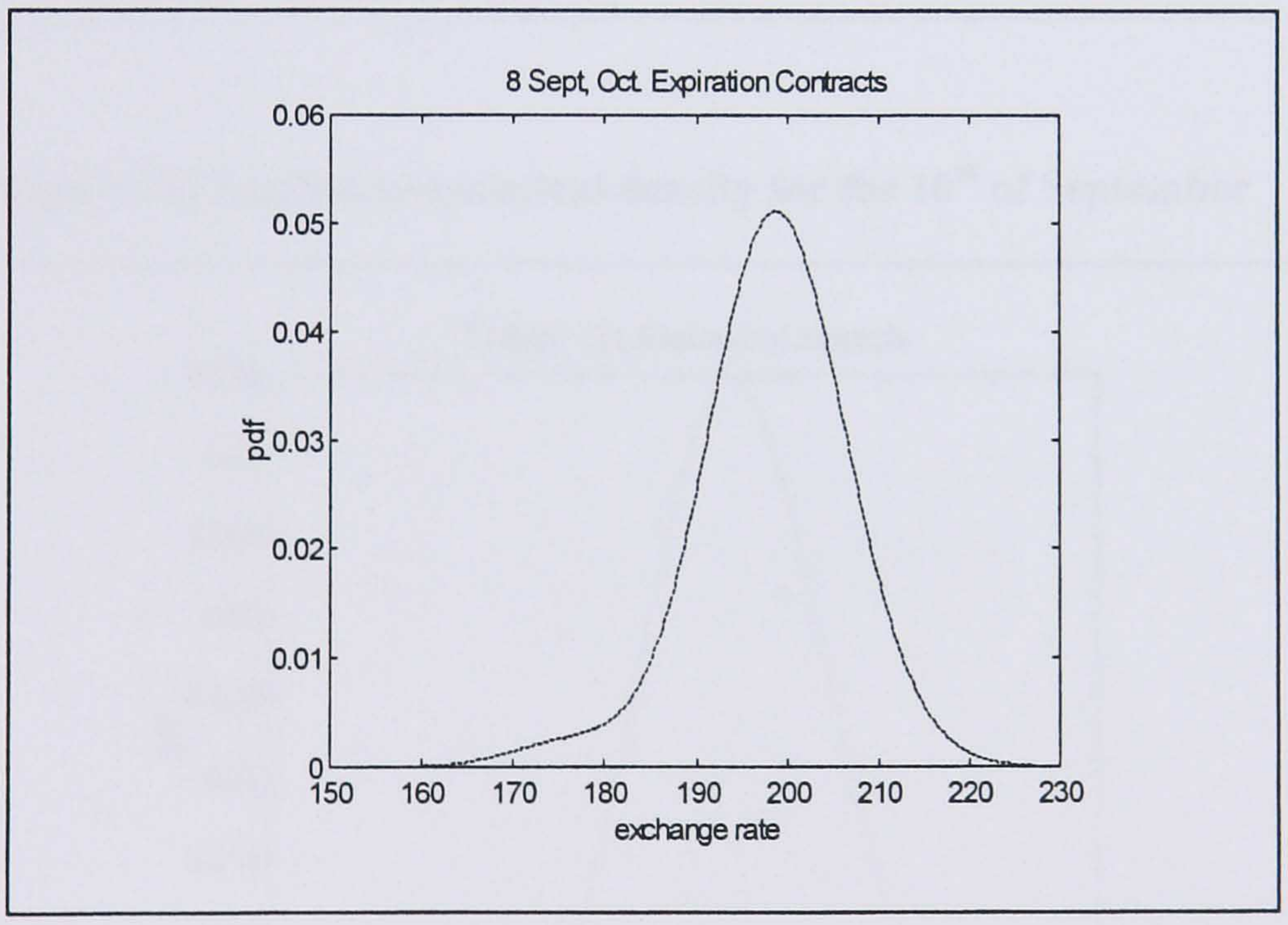


Figure 9: Implied risk-neutral density for the 9th of September

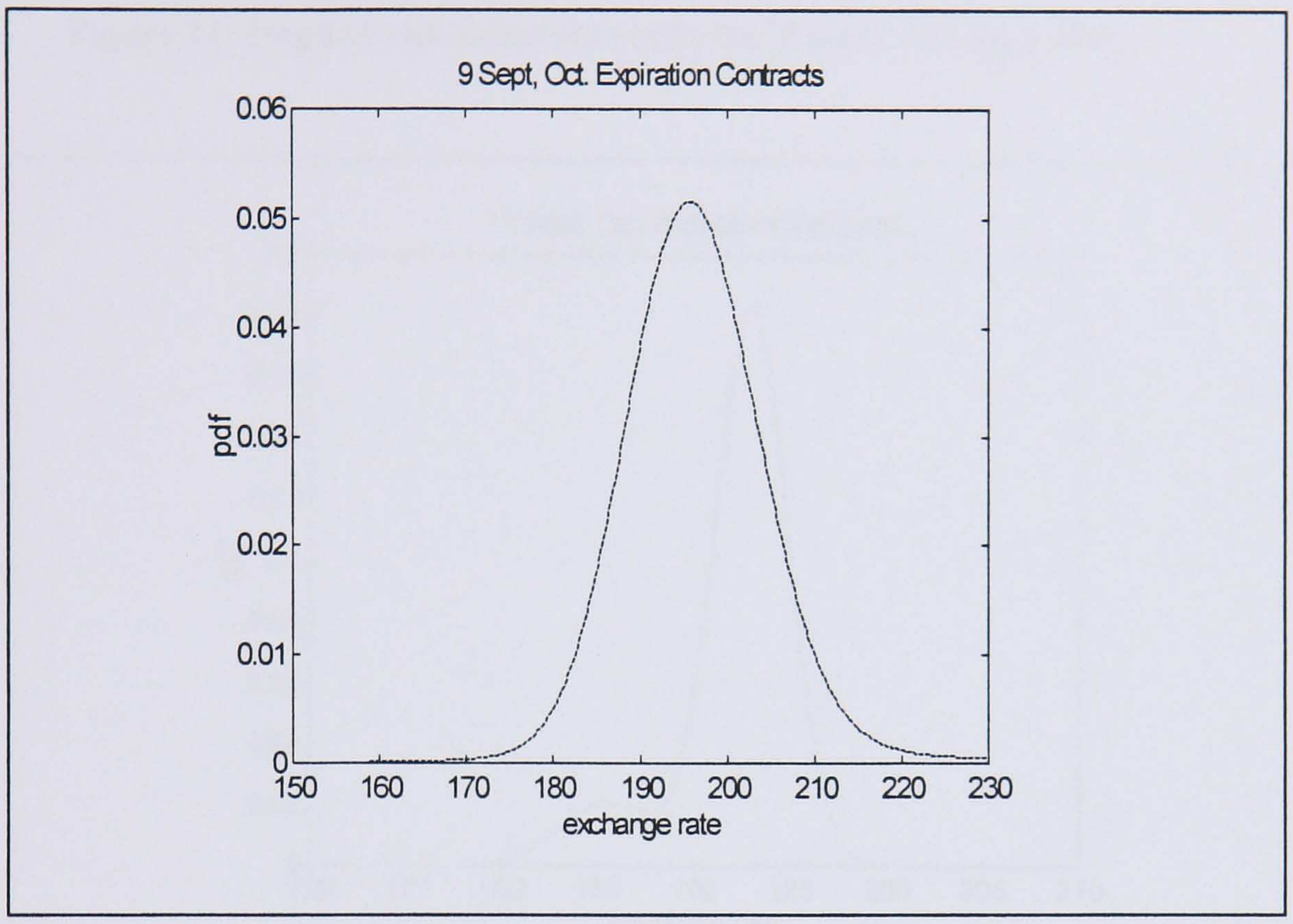


Figure 10: Implied risk-neutral density for the 10th of September

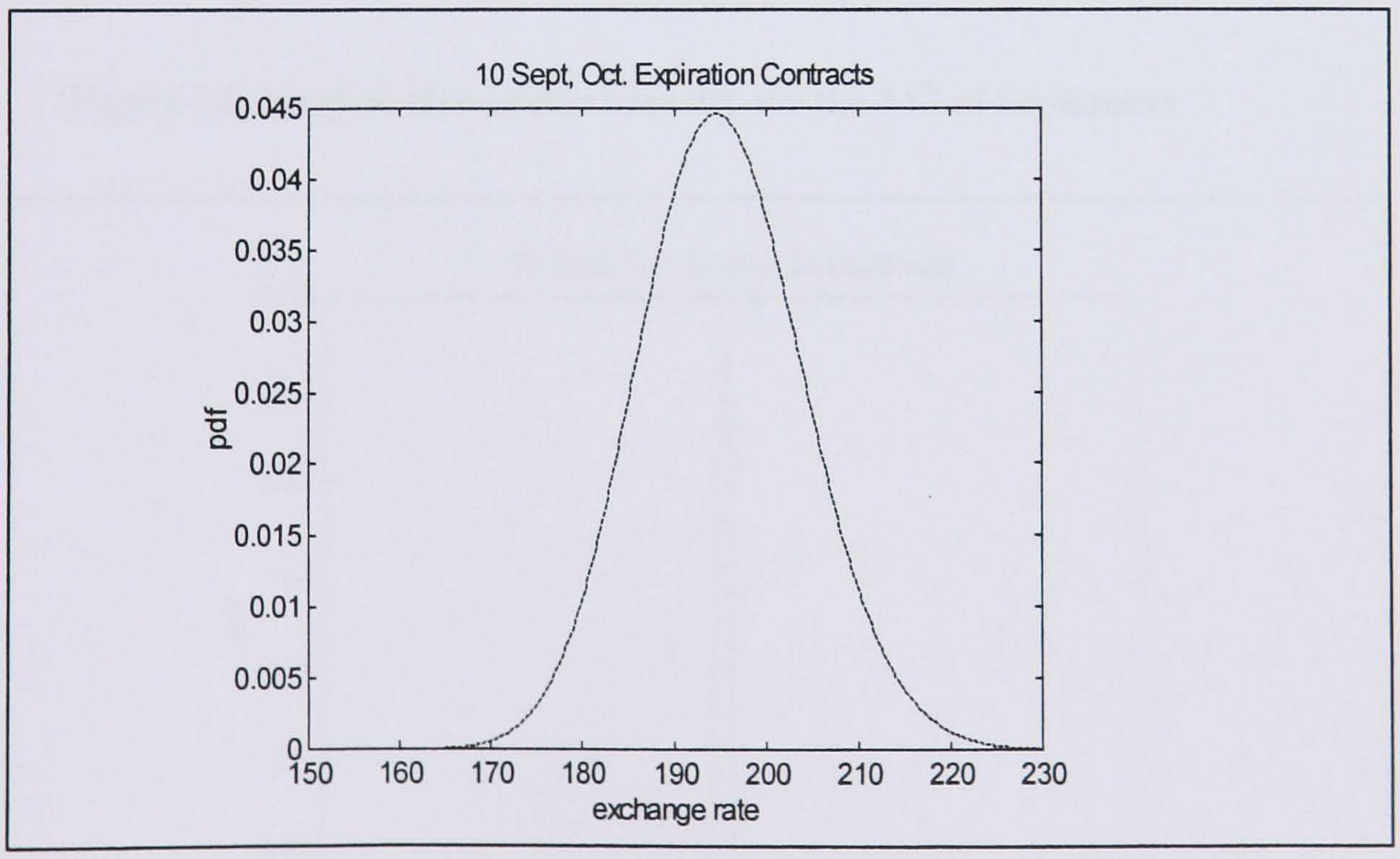


Figure 11: Implied risk-neutral density for the 11th of September

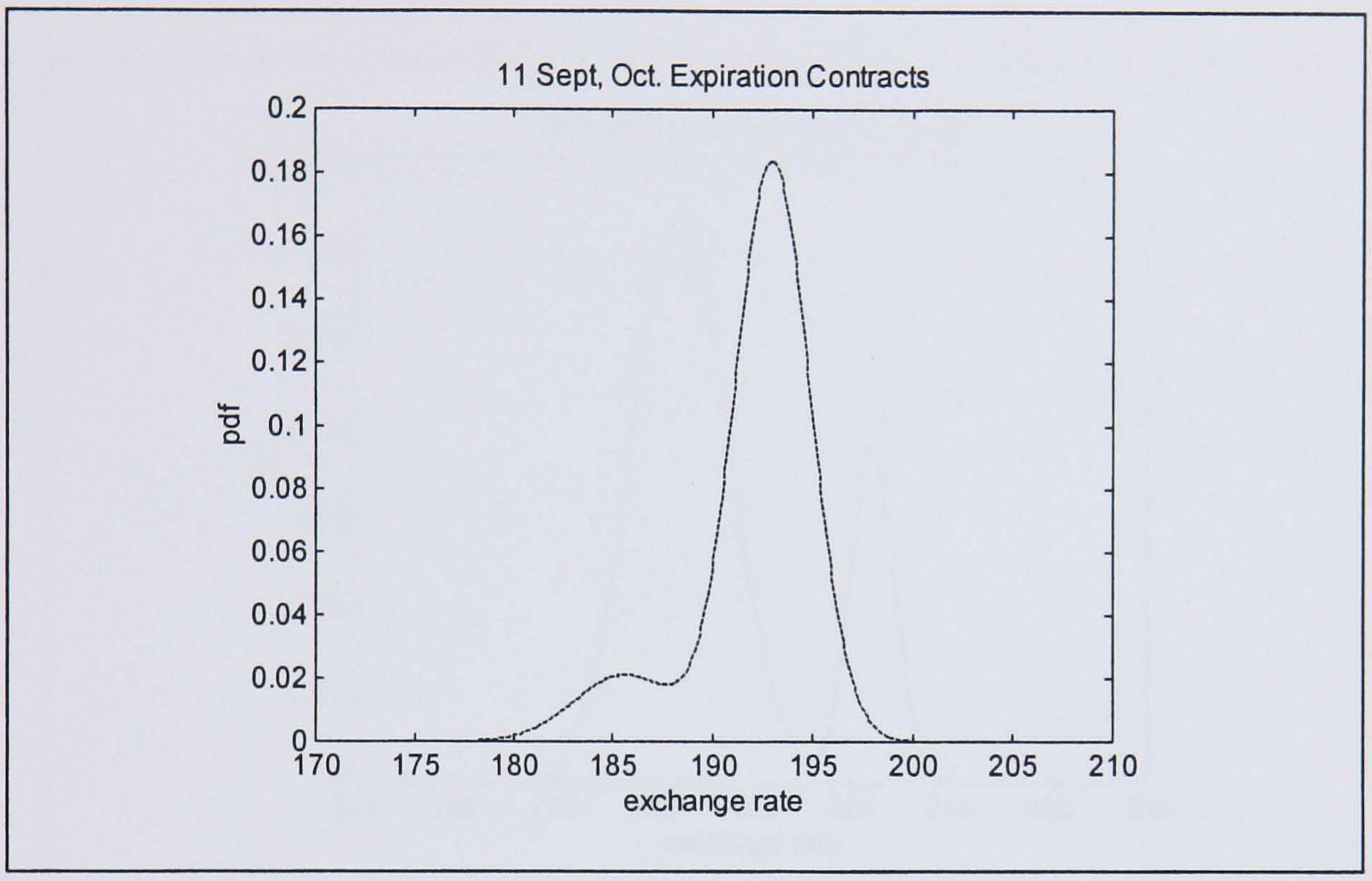


Figure 12: Implied risk-neutral density for the 14th of September

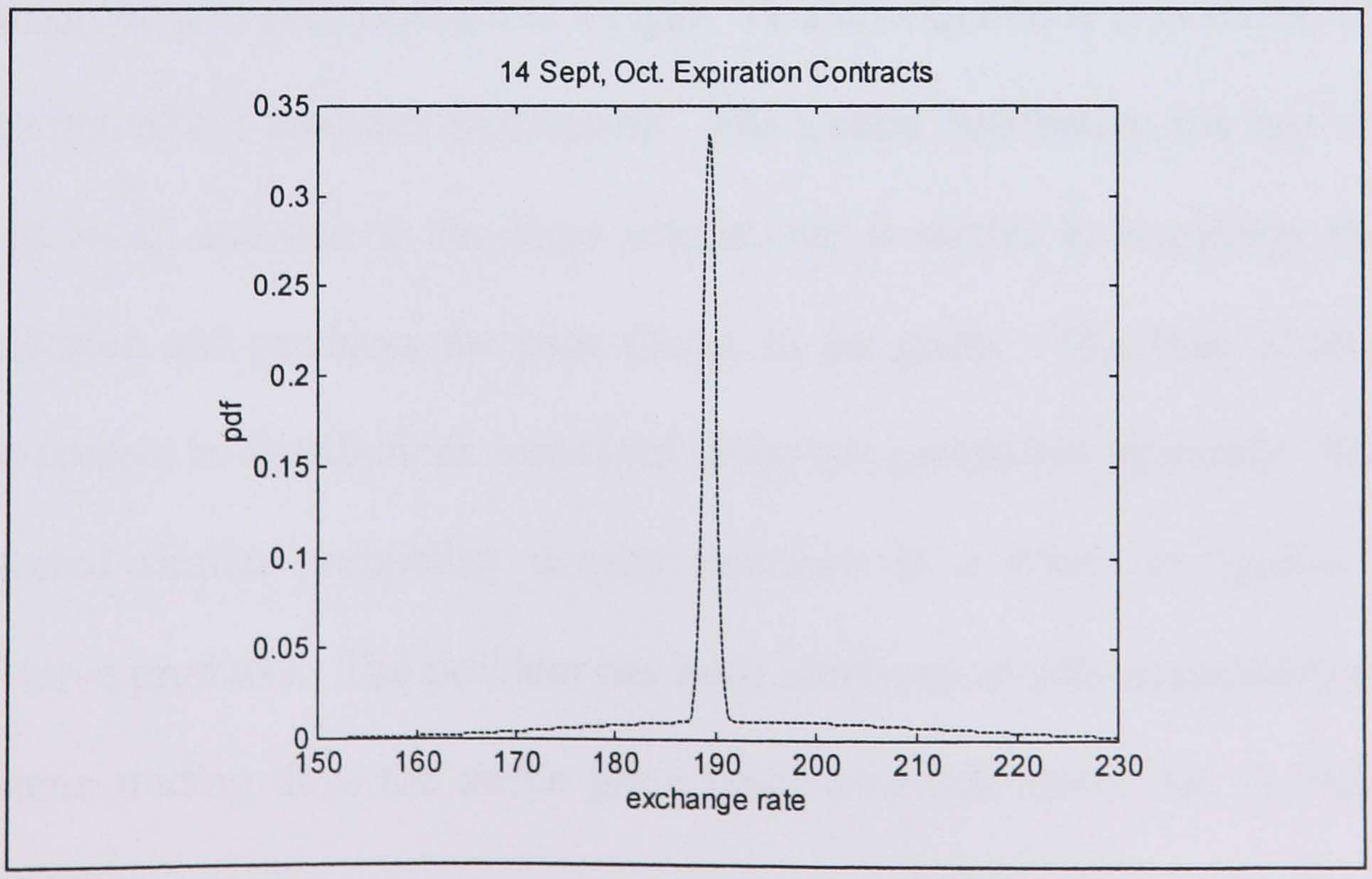
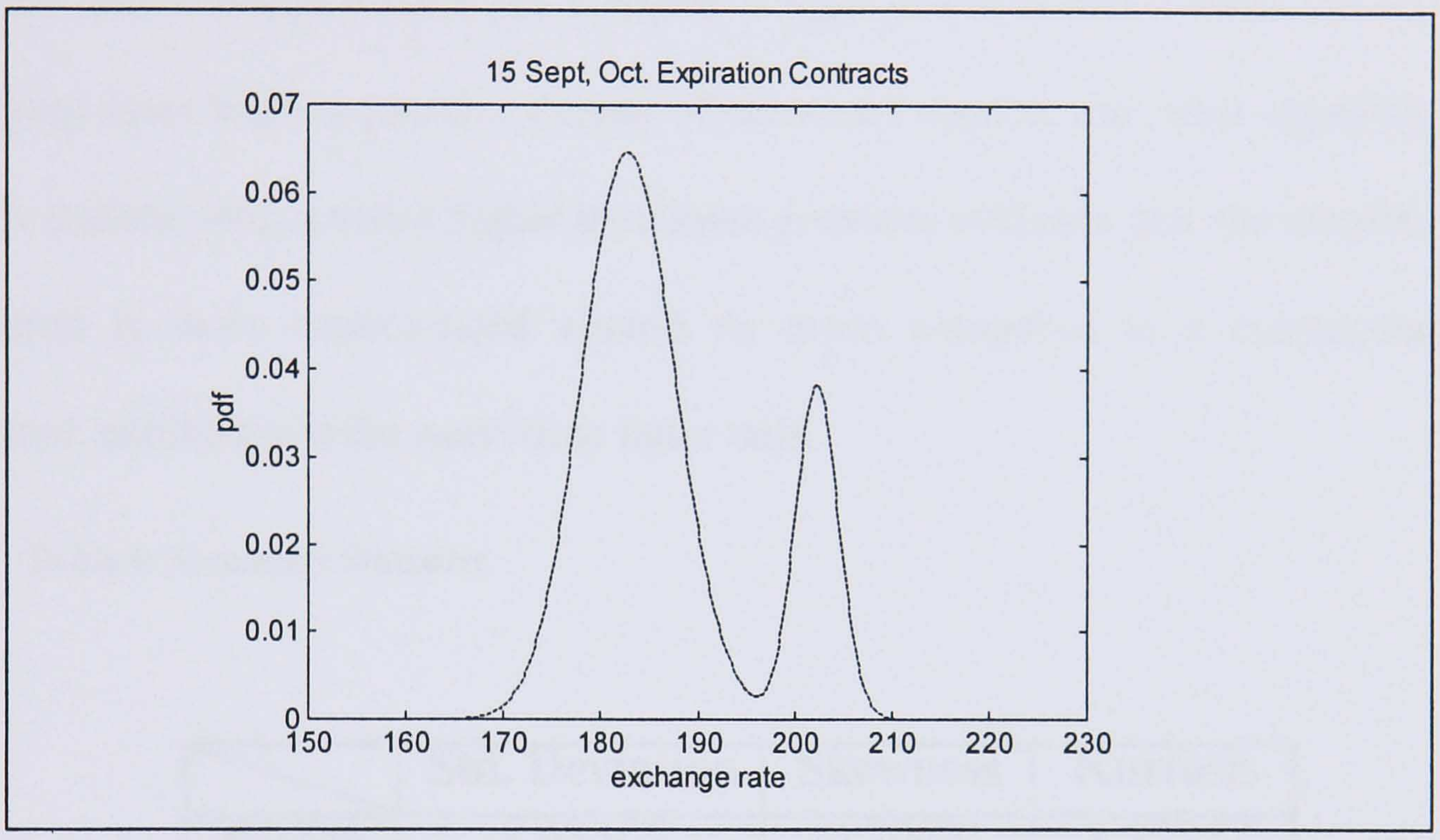


Figure 13: : Implied risk-neutral density for the 15th of September



The graph of the 14th of September is highly irregular since the two constituent lognormal distributions exhibit very different characteristics. The one has a very high variance but it is attributed a low weight. This distribution is responsible for the wide dispersion of the mixtures distribution. The second distribution has low variance but a high mean and due to the large weight that it carries it dominates the resulting distribution and produces the peak shown in the graph. This type of irregularity is often present in distributions recovered using this parametric approach. Bahra (1997) recovered similar probability density functions in a study on options on LIFFE derivative products. The problem has been attributed to data irregularities. Namely, on some trading days the strike price range available could not provide adequate information for the recovery of the full distribution. This has as a direct consequence, that any inference made based on such a distribution might be inaccurate.

Summary statistics estimated from the implied risk-neutral density functions, are presented in the table below. Standard deviation is a measure of dispersion, skewness is a measure of symmetry and kurtosis a measure of fat-tailness. Negative skewness indicates that traders put a higher weight to downward movements of the underlying asset while a positive degree of skewness implies the exact opposite. A kurtosis statistic with a value higher than three provides evidence that the distribution considered is more concentrated around its mean compared to a corresponding lognormal, exhibiting at the same time fatter tails.

Table I: Summary Statistics

	Std. Deviation	Skewness	Kurtosis
04-Sep	8.68407	-0.20071	3.52227
08-Sep	8.66352	-0.47931	4.0603
09-Sep	8.30936	1.32228	8.29378
10-Sep	8.95501	0.15107	3.05329
11-Sep	3.22644	-1.27647	4.80622
14-Sep	12.56339	2.60064	10.25824
15-Sep	9.10095	0.69668	2.38394

Some other useful distributional statistics are the 25, 50 and 75 percent quartiles, denoted by the symbol Q_d (d stands for the percentage quartile). They represent the value such that the probability mass of the implied risk-neutral density function below this value is equal to 25, 50 or 75 percent respectively. It is:

$$P(S \leq Q_d) = d$$

Their usefulness stems from the fact that they provide an indication of how evenly the probability mass of the implied density is distributed. The fifty percent quartile Q_{50} has the special property that half of the "distribution" lies on each side of it and it is also called the median of the distribution. Two additional statistics are reported in Table II below, the mean and the mode.

The latter is the most probable value that the variable might take or in other words it is the maximum value of the density function.

$$\mathrm{mod}\, e = \max_{S} f(S)$$

Table II: Mean, mode and quartiles

	Q_{25}	Q_{50}	Q_{75}	Mean	Mode
04-Sep	190.9427	196.5686	202.1502	196.4092	196.6315
08-Sep	193.2168	196.5686	203.8292	198.1891	198.7141
09-Sep	191.0067	196.1519	201.5063	195.8463	195.7376
10-Sep	189.0322	194.9786	201.1121	195.1836	194.5679
11-Sep	190.9645	192.652	194.0665	192.0222	192.9951
14-Sep	188.7612	189.6109	195.4257	190.6281	189.4169
15-Sep	180.6004	184.5779	191.2872	187.0997	182.7461

The statistics available compose a clear picture about the events that preceded the sterling pound's exit from the Exchange Rate Mechanism on September 16th 1992. The market seemed not to have a definite view as to how the \$/£ exchange rate was to behave over the next period of time, during the first days of week starting on the 8th of September. Even though skewness coefficients for the 4th and the 8th of September are negative the quartiles or the mode of the distribution do not show any signs of a depreciation fear. On that Friday the 11th of September, though, the first shadows are being casted on the sustainability of the current exchange rate regime. A strong bimodal pattern becomes present in the implied distribution with a left hump, something, that is also depicted in the strongly negative skewness coefficient estimated for that day. Furthermore, the median and the 75 percent quartile exhibit a significant decline, indicating a transfer of probability mass towards the left of the distribution. The distribution is highly skewed on that day and has a very small implied standard deviation, which suggests that investors were relatively certain that a depreciation was imminent.

On September 14th the British pound did indeed depreciate by –1.68% and on September 15th the situation was almost clear. The 25% percent quartile exhibited a 10 cents drop relatively to its value on the 11th and the median followed closely. However, the skewness coefficient had a positive sign, which is counter-intuitive. Observing the graph of the implied risk-neutral density function for that day, though, we can conclude that this a more of a mathematical feature, rather than a feature related to market sentiment. The two-humped mixtures distribution recovered for the 15th of September suggests that investors had well abandoned the idea of an expensive pound placing the most probability weight on a depreciating scenario. It is in effect the transformation of the small hump present in the September 11th implied graph to the main hump present in the graph of the distribution for September 15th. Nevertheless, mathematically the main body of the mixtures function now is on the left of the distribution and the smaller hump on the right gives the estimated skewness the positive sign.

Following Mizrach (1996), the risk-neutral probability of a depreciation of three percent or more for the sterling pound was additionally computed. It should be noted that the three percent depreciation is measured with respect to the risk-neutral mean and not with respect to the actual value of the underlying asset. The estimation results are shown in Table III:

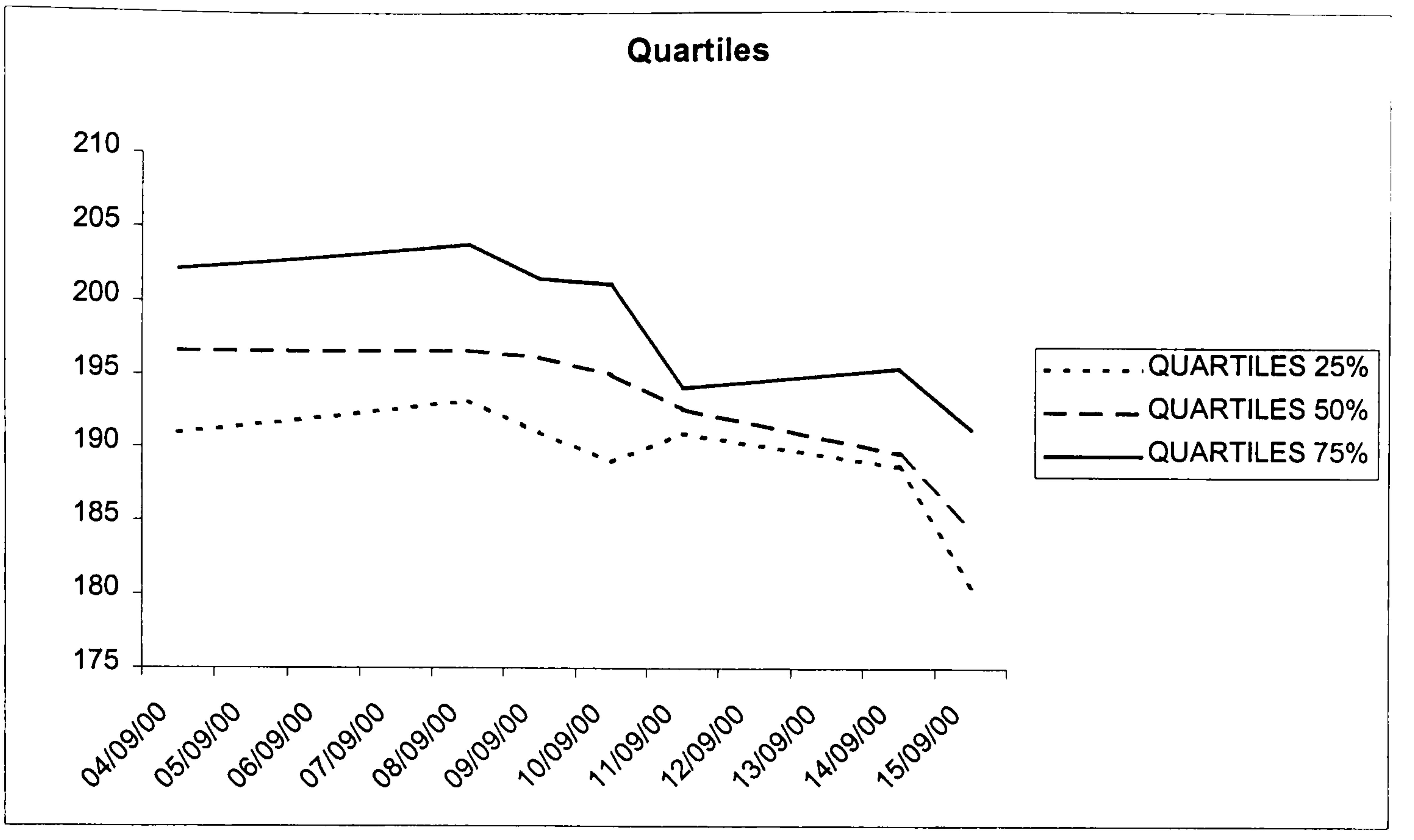
Table III: Risk-neutral probabilities of a three percent or more depreciation of the $/£

	Prob.-3%
04-Sep	0.220435
08-Sep	0.21414
09-Sep	0.20835
10-Sep	0.20631
11-Sep	0.08187
14-Sep	0.13982
15-Sep	0.30378

The results presented in Table III above clearly indicate that on the day preceding the pound's exit from the ERM the probability of a three percent or more decline in the $/£ exchange rate peaked, taking a value that was almost 38% higher than the highest value it had taken thus far. This is in agreement with Mizrach (1996) who finds that this probability does become significant but only in the last day before the 16th of September.

The argument that there has been a significant re-evaluation of traders' expectations on the 15th of September is further supported by Figure 14. On this graph the three main quartiles are plotted. As it is clearly illustrated there is a significant shift in all three quartiles on that day, which signifies that traders became almost certain that the sterling pound could not maintain its strength for much longer.

Figure 14: Quartiles Plot



Additional evidence presented in Table IV, originating from trading data, concerning the volume of option contracts being traded prior to the 16th of September, illustrate that investors had been trading consistently more in put rather in call options. This provides us with further insight on the bearish market sentiment for the British pound, prevailing at that time.

Table IV: Trading Volume data

	Available Strikes		No. of Transactions	
	Calls	Puts	Calls	Puts
04-Sep	12	18	36	46
08-Sep	10	18	21	45
09-Sep	18	22	44	57
10-Sep	19	27	64	76
11-Sep	20	32	105	124
14-Sep	14	32	32	121
15-Sep	20	23	50	66

5.4 CONCLUSIONS

In this chapter an analysis of the event of the pound's exit from the ERM is performed through the prism of a mixture of two lognormal distributions as a choice for the RND of the \$/£ exchange rate. The results of the analysis suggest that markets were not taken completely by surprise at the time.

6 CHAPTER SIX: ASIAN OPTIONS

In the previous Chapter the primary purpose that implied methodologies serve was examined. That is, the extraction of information embedded in option prices for the recovery of the market sentiment. Policy makers then, can use this information in order to plan or adjust their policy or alternatively in order to analyse the market reaction to the course of action already taken.

As it was mentioned earlier in the text the second use of implied models is the pricing of exotic products in a consistent fashion with their simple vanilla counterparts. Achieving consistent pricing of exotic products with vanilla traded options implies that when the simple call and put options are used to hedge the more complicated OTC structure then even if the products included in the hedge may not be correctly priced, the hedge, however, should remain valid and possibly unaffected. This should happen because the exotic and the vanilla instruments are priced using the same distributional assumptions and the same set of information and they should be expected to deviate from the corresponding "correct" prices by a similar amount.

In this chapter and in the following I will examine the latter use of implied distributions. Hence, I will exploit recovered information from observed option prices in order to price exotic options in a market consistent way. I will apply an implied model for the pricing of Arithmetic, Geometric and Average options as well as for Basket options.

6.1 Asian Options

This chapter develops an implied pricing methodology for Asian options. Asian options belong to the family of exotic path dependent options since their payoff depends on the average (arithmetic or geometric) of the underlying asset over some prescribed period of time.

It is said that the name Asian options was coined by employees of Bankers Trust, which sold these options to Japanese firms that wanted to hedge their foreign currency exposure. These firms used these options because their annual reports are also based on average exchange rates over the year.

Asian options have always been extensively used. Since the mid 80's the average rate, or Asian, options have become one of the most popular exotic option products that are mainly traded in the Over the Counter (OTC) market. Already by the end of the 70's average rate options were part of the so-called commodity linked bonds. One of the first commodity linked bonds was the Mexican Petrobond issued in 1977 and featuring redemption at 25-day-interval averages. In May 1985 the Dutch company Oranje Nassau issued bonds with a maturity of eight years in local currency; the redemption value was defined as the maximum of the average price of 10.5 barrels of Brent Blent oil over the last year of the contract and the face value of the bond. The average was based on monthly prices and when the bonds were issued, the value of the amount of oil was equal to the face value. Hence, these were the at-the-money options. By adding this feature to the contract, the company could set the coupon rate at one percent below the prevailing market rate.

The company had two reasons for using an average rate option in the contract. First, its profits were correlated with the oil price and for high oil prices it would be no problem to pay the high redemption value out of their profits. For low oil prices the options would expire worthless. However, profits would only be large enough if the oil price was high during a substantial part of the final year and not only at the maturity date. The average rate option would end in-the-money only in the first case. This is still the main reason for the use of average rate options. The second reason at that time was that the company feared oil price manipulation near the maturity date of the bonds. Since the payoff of Asian options depends on the underlying asset's value over a longer time period price manipulation becomes very difficult if not impossible.

Within the Black and Scholes framework Geometric Asian options admit closed form solution, whereas the pricing of Arithmetic Asian options involves the evaluation of the distribution of the sum of lognormal random variables, which is not amenable to a closed form distributional expression. Hence, a number of numerical approaches and analytical approximations have been proposed.

Kemna and Vorst (1990) in the first paper that was published on Asian options proposed a Monte-Carlo simulation method with variance reduction technique. They used as a control variate the corresponding Geometric average option to enhance the computational speed.

Caverhill and Clelow (1990) proposed a solution for the AA option problem by taking a fast Fourier transform of the convolutions of the densities of the individual components that constitute the value of an AA option in order to approximate the density function of the sum of the components. They, then, integrate the payoff function against the convoluted density.

Rogers and Shi (1995) dealt with the problem through a solution of a parabolic PDE in two variables, considering Asian options in continuous time. They also propose an alternative method providing very accurate lower bounds of the price of an Asian option. A PDE approach was employed by Alziary, Decamps and Koehl (1996) with an application to both "fixed strike" and "floating strike" Asian options. Zvan, Forsyth and Vetzal (1997) explore the robustness of the numerical methods used to solve the Partial Differential Equations for the valuation of average rate options using techniques from the field of computational fluid dynamics.

Yor (1992) and Geman and Yor (1993) as well as Rogers and Shi (1990) concentrate on continuous Asian options. They make use of the theory of Bessel processes to find explicit formulas involving complex integrals for the values of continuous Asian options. To calculate prices, one has to evaluate these integrals numerically. A similar technique was also used by Geman and Eydeland (1995).

Bouaziz, Briys and Crouhy (1994), derive a closed-form solution to the pricing problem of Asian options, using a simple linearisation procedure. They apply this formula for the valuation of European style vanilla Asian contracts and for forward-starting ones. Curran (1994) attempted to price Asian options by conditioning on the Geometric mean employing numerical integration. Zhang (1995) proposed a second order approximation to the AA mean with the Geometric average one by approximating the general mean function: $M(\gamma)=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\gamma}\right)^{1/\gamma}$ with its derivative according to the mean value theorem. He made the assumption that the sum of lognormal variables is also approximately lognormally distributed.

Levy (1992), assumed that the distribution of the sum of lognormal variables can be well approximated by another lognormal variable. He found the mean and variance of this new approximating distribution and provided a closed form solution for the AA. Turnbull and Wakeman (1991) expanded Levy's method by using an Edgeworth series expansion to approximate the distribution of the sum of the lognormal variables. They found a fourth order approximation of the distribution of the sum of the n lognormal variables. They observed that moments of the sum could easily be estimated, and obtained a very good approximation for the Asian option price. A very similar approach is taken by Ritchken, Sankarasubramanian and Vijh (1993), who employ Edgeworth series expansions in order to derive approximate prices for average rate and lookback options.

These attempts, however, all share a common characteristic. They all assume that the underlying asset's price follows a Geometric Brownian Motion (GBM) process, or equivalently, that the underlying asset's terminal distribution is lognormal.

In this chapter a new valuation framework for Asian options is presented. The dynamics of the underlying asset are assumed to be described by a Jump Diffusion model with a Bernoulli distributed jump component. A closed form solution is derived for Geometric Asian option and in so far as the sum of lognormal variables can be approximated by another lognormal variable[1], a closed form solution is derived for

[1] See Fenton (1960), Janos (1970), and Barakat (1976).

Arithmetic Asian options as well. Given that the defining parameters of the underlying asset's risk neutral stochastic process can be entirely recovered from observed vanilla option prices, the pricing of Asian options in the setting adopted in this chapter is an implied valuation problem.

Furthermore, if the stochastic process of the underlying asset can be described by a Bernoulli jump diffusion stochastic differential equation, the terminal risk neutral density of the asset can be found to be a 2-component mixtures lognormal distribution. Both Geometric and Arithmetic Asian options are priced within this framework. In order to determine the validity of the proposed method, Monte Carlo simulation is performed and a test is made whether the computed option prices are in accordance with the ones postulated by the distributional assumptions made.

6.2 The Bernoulli Jump Diffusion Framework

In a Black and Scholes framework the underlying asset follows a Geometric Brownian Motion (GBM) process,

$$dS = \mu S dt + \sigma S dZ \tag{1}$$

, which implies that the terminal time T, risk neutral distribution (RND) of S_T, $q(S_T)$ is lognormal with parameters α, b, i.e:

$$q(S_T) = \frac{1}{S_T b\sqrt{2\pi}} e^{\frac{-[\ln(S_T)-a)]^2}{2b^2}} \tag{2}$$

for $a = \ln S + (\mu - 0.5\sigma^2)\tau, \quad b = \sigma\sqrt{\tau}$

Ball and Torous (1983, 1985) examined an alternative parameterisation for the stochastic process of the underlying that could take into account possible asymmetries present in the return process. They suggested a jump diffusion process with a Bernoulli distributed jump component. The stochastic process they suggested is shown in the equation below:

$$\frac{dS}{S} = adt + \sigma dZ + dq$$

where a is the expected return on the stock and dq is the Bernoulli distributed jump component.

This initial rather simplistic version of the Bernoulli jump diffusion model was further developed by Malz (1996). He used a similar SDE to extract realignment probabilities from option prices and showed how it can be utilized in order to extract implied distributions when only little information is available. The Stochastic Differential Equation (SDE) that describes the dynamics of the underlying asset in this case is shown below:

$$\frac{dS}{S} = (r - d - \lambda k)dt + \sigma_w dW_t + kdq \qquad \textbf{(3)}$$

,where r is the interest rate, d is the dividend yield, σ_W is the instantaneous variance of the underlying asset returns conditional on no jumps occurring, k is the percentage jump conditional on the Bernoulli event occurring and λ is the intensity of the Bernoulli distribution. The jump component was assumed to be deterministic and not random.

Within this framework the price of a European, say call option is derived in the following manner.

$$C = e^{-rT}\,\mathrm{Pr}ob(J_T)E\left[(S_T - X)^+ / J_T\right] + e^{-rT}\,\mathrm{Pr}ob(\overline{J_T})E\left[(S_T - X)^+ / \overline{J_T}\right]$$

where $J_T = \{\text{One jump occurs between now and time T}\}$.

$$\begin{aligned} C = (1-\lambda T)\cdot e^{-rT}&\left[\frac{Fut}{1+\lambda kT}N(d_1) - XN(d_2)\right] + \\ &\lambda T \cdot e^{-rT}\left[\frac{Fut}{1+\lambda kT}(1+k)N(d_3) - XN(d_4)\right] \end{aligned} \tag{4}$$

with $d_1 = \dfrac{\ln(Fut/X) - \ln(1+k\lambda T) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ $\quad d_2 = d_2 - \sigma\sqrt{T}$

and

$$d_3 = \frac{\ln(Fut/X) - \ln(1+k\lambda T) + \ln(1+k) + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}} \quad d_4 = d_3 - \sigma\sqrt{T}$$

In equation (4) the Futures price of the underlying is weighted by the factor $1+\lambda kT$ the expected value of the deterministic jump. As Malz (1996) explains "...Intuitively, the sterling-mark (the underlying asset in Malz's study) must already have appreciated by say, 5 percent, to reflect a jump with an expected value of 5 percent. Otherwise the weighted average price of the zero-jump and one-jump future spot rates would not equal the current forward rate and the option price would not be risk-neutral...."

Jondeau and Rockinger (1999) and McManus (1999) explore the potential of this jump-diffusion framework for the recovery of implied distributions.

However, the methodology that is suggested in this study has an important difference from the one followed by Malz (1996), Jondeau and Rockinger (1999) and McManus (1999). The former researchers assumed that even though the jump process had a Bernoulli distributed random event, the size of the jump component on the contrary was deterministic and constant. This study generalizes the above described framework by assuming that the jump component is stochastic and moreover, normally distributed. The specifications of the new distributional assumptions are presented below.

$$\ln(1+k) \sim N(\gamma - \tfrac{1}{2}\delta^2, \delta^2) \equiv N(\gamma', \delta^2), E(k) \equiv \bar{k} = e^{\gamma} - 1$$

These assumptions are in accordance with Bates (1991) and (1996a) where the same specifications are adopted for the Poisson jump component that is used in that paper.

The call option pricing formulae under these assumptions now become:

$$\begin{aligned} &= (1-\lambda T)\cdot e^{-rT}\left[\frac{Fut}{1+\lambda E[k]T}N(d_1) - XN(d_2)\right] + \\ &\qquad \lambda T\cdot e^{-rT}\left[\frac{Fut}{1+\lambda E[k]T}(1+E[k])N(d_3) - XN(d_4)\right] \end{aligned} \tag{5}$$

With

$$d_1 = \frac{\ln(Fut/X) - \ln(1+E[k]\lambda T) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad d_2 = d_2 - \sigma\sqrt{T}$$

and

$$d_3 = \frac{\ln(Fut/X) - \ln(1+E[k]\lambda T) + \ln(1+E[k]) + \frac{1}{2}(\sigma^2 T + \delta^2)}{\sqrt{\sigma^2 T + \delta^2}} \qquad d_4 = d_3 - \sqrt{\sigma^2 T + \delta^2}$$

Similar differences in the two methodologies are present when implied distributions are examined. It is a known mathematical fact that a specific stochastic process results to a single terminal distribution for the underlying asset, while the reverse does not hold. That is a terminal distribution can be consistent with an arbitrary number of stochastic processes. Within the Black and Scholes framework the underlying asset follows a GBM which results in a single lognormal distribution for the terminal risk neutral density as it is illustrated in (2).

Alternatively, when the stochastic process describing the movement of the underlying is a Bernoulli jump diffusion process with a jump component that has a deterministic size, then the terminal risk neutral distribution can be mathematically proven to be a mixture of two lognormal distributions. The cumulative distribution function of the mixtures distribution is represented by the function below

$$prob(S_T \leq X) = (1-\lambda T)\cdot\Phi\left[\frac{\ln\left(\frac{X}{Fut}\right)+(\ln(1+\lambda k)+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right] + \lambda T\cdot\Phi\left[\frac{\ln\left(\frac{X}{Fut}\right)-\ln(1+k)+(\ln(1+\lambda k)+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right] \qquad \textbf{(6)}$$

where $\Phi[..]$ is the standard normal cumulative distribution function and $Fut = S_t e^{(r-r^*)T}$ is the forward rate. It is implicitly assumed that r is the risk-free interest rate and r^* is the dividend yield of the underlying asset.

The general form of the distributional function of a density belonging in the family of mixture of two lognormal distributions is

$$prob(S_T \le X) = (1-\theta)\Phi\left[\frac{\ln X - a_1}{b_1}\right] + \theta\Phi\left[\frac{\ln X - a_2}{b_2}\right] \tag{7}$$

Observing (6) and (7) we can infer the following parameter correspondence between the parameters of the stochastic process and those of a mixture of two lognormals.

Mixture of two lognormals	*Bernoulli Jump Diffusion model*
$\theta = \lambda\tau$	$\lambda = \theta / \tau$
$\beta_1 = \sigma\sqrt{\tau} = \beta_2$	$\sigma = \beta_i / \sqrt{\tau},\ i = 1,2$
$a_1 = \ln\frac{Se^{(r-d)\tau}}{1+k\lambda\tau} - \frac{1}{2}\sigma^2\tau$	$k = e^{a_2 - a_1} - 1$
$a_2 = a_1 + \ln(1+k)$	

Table I: Parameter correspondence with deterministic jump size

It is obvious from Table I, the variances of the two components of the mixtures distribution are found to be equal. This does not come as a surprise the underlying asset was assumed to have a jump component that is non-deterministic.

By allowing the jump size to vary stochastically the risk neutral density in equation (6) is transformed into the following more general form.

$$prob(S_T \le X) = (1-\lambda T)\cdot\Phi\left[\frac{\ln\left(\frac{X}{Fut}\right)+(\ln(1+\lambda k)+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right]$$

$$+\lambda T\cdot\Phi\left[\frac{\ln\left(\frac{X}{Fut}\right)-\ln(1+k)+(\ln(1+\lambda k)+\frac{\sigma^2 T+\delta^2}{2})T}{\sqrt{\sigma^2 T+\delta^2}}\right] \quad \textbf{(8)}$$

The parameter correspondence between the mixture of lognormals distribution and the Bernoulli Jump diffusion model will also be altered.

Mixture of two lognormals	*Bernoulli Jump Diffusion model*
$\theta = \lambda\tau$	$\lambda = \theta/\tau$
$\beta_1 = \sigma\sqrt{\tau}, \beta_2 = \sqrt{\sigma^2\tau+\delta^2}$	$\sigma = \beta_i/\sqrt{\tau}$
$a_1 = \ln\frac{Se^{(r-d)\tau}}{1+k\lambda\tau} - \frac{1}{2}\sigma^2\tau$	$\delta^2 = (\beta_1^2 - \beta_2^2)/\tau$
$a_2 = a_1 + \ln(1+k)$	$k = e^{a_2-a_1} - 1$

Table II: Parameter correspondence with stochastic jump size

Melick and Thomas (1996), Bahra (1996), Dinenis, Flamouris and Hatgioannides (1999) and others have shown how the parameters of the distribution can be recovered from observed option prices. However, the mixture of lognormals methodology cannot be used for the extraction of the parameters of the Bernoulli jump diffusion parameters.

Instead the approach described in Malz's (1996) should be adopted . This approach is also used in Jondeau and Rockinger (1999) and in McManus (1999).

As the above researchers point out care should be taken in the parameter recovery procedure. Malz (1996) emphasizes the difficulties in the recovery of the exact parameter values in (8) of the parameters k and λ as they appear in pairs. He proposes a two-step algorithm for parameter values in order to overcome this problem and recover accurate estimates of these two parameters. In this chapter and in the following, it will be assumed that the parameters of the Bernoulli distributed stochastic process can be recovered using this methodology and the issue will not be addressed again.

The mathematical correspondence between the mixture of lognormal distributions and the Bernoulli jump diffusion process further led me, as previously mentioned, to consider the mixture of two lognormal distributions as the parametric form assumption for the recovery of implied distributions in the **Part 1** of the thesis. This probably unique characteristic that this specific distributional form possesses makes it an ideal choice for a risk-neutral density since both uses of implied distributions can be explored and exploited.

6.3 Asian Option Valuation

The payoff of an Asian call option is defined as the maximum of the average value of prices of the underlying asset observed over a period of time minus the strike price. To express this payoff for both call and put options a more compact form is used:

$$AS = \max[wAV - wK, 0] \qquad \textbf{(9)}$$

where w is 1 or −1 depending on if a call or a put option is priced respectively, K is the strike price and AV is the average of the observed underlying asset's values. There are two kinds of Asian options based on the way the averaging is being made. There are the arithmetic Asian options when AV is the arithmetic average of prices:

$$AV_A(n) = \frac{1}{n}\sum_{i=1}^{n} S_i \tag{10}$$

and the Geometric Asian options defined when AV is the geometric average of prices:

$$AV_G(n) = \left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{n}} \tag{11}$$

where S_i is the ith observation of the underlying asset's price.

If the GBM assumption is made for the stochastic process that governs the movement of the underlying asset, then the underlying asset's price at any generic future time T is given by:

$$S(T) = S\,exp\left[\left(r - d - \frac{1}{2}\sigma^2\right)T + \sigma Z(T)\right] \tag{12}$$

Geometric Asian options will be first dealt with since given (12) it can be easily seen that by taking logarithms we transform the product in equation (11) to a simple summation of normal variables. In such a context, closed form solutions can be found; see Kemna and Vorst (1990).

The payoff of a European Geometric Asian option can be expressed as follows:

$$GA = \max[wAV_G(n) - wK, 0] \tag{13}$$

where w is either 1 or –1 depending on whether the option under consideration is a cal or a put option respectively. K is the strike price and $AV_G(n)$ is given from (11), the Geometric average of n discretely observed underlying asset's prices. To end up with a closed form formula we take expectations in the previous formula and the value of the option is given from:

$$OP = e^{-rT} E(\max[wAV_G(n) - wK, 0])$$

6.4 Geometric Asian Option Valuation With A Bernoulli Jump Diffusion Model

To price a Geometric Asian option using this distribution it will helpful to express the asset price in a way similar to (12). Following Merton (1976), we write:

$$S(T) = \frac{S}{1 + \lambda T E[k]} exp((r - d - \tfrac{1}{2}\sigma^2)T + \sigma Z(T))Y(T) \qquad \textbf{(14)}$$

,where $Y(T)$ is a variable that is either equal to one with probability (1-θ) or it is equal to Y with probability θ. The logarithm of the variable Y follows a normal distribution: $\ln Y \sim N(\mu_Y, \sigma_Y^2)$. Intuitively in the time period between now and future time T we expect for the underlying to exhibit either no jumps with probability (1-θ) or one jump with probability θ.

The distribution of $AV_G(n)$ needs to be known in order to calculate the price of a European Geometric Asian option. The same tactic will be followed, as with the GBM case and the logarithm of the Geometric Average will be considered since it exhibits properties very close to those of a normal distribution, making calculations possible. Thus, we are interested in the distribution of the logarithm of GA(n).

Let us express the time of the jump as τ. Dividing the time period into n subintervals of equal length h, at the edges of which an observation is made for the price of the Asian option, and using (14) we may write:

$$\begin{aligned}
&\ln(S(t+h)) = \ln S + ah + \sigma Z_1^n + \ln Y \cdot I_{\tau<t+h} \\
&\ln(S(t+2h)) = \ln S + a2h + \sigma(Z_1^n + Z_2^n) + \ln Y \cdot I_{\tau<t+2h} \\
&\vdots
\end{aligned} \tag{15}$$

,where Z_j^n is the incremental information arriving at time $t+jh$ and $I_{\tau<t+jh}$ is an indicator function that takes the value of 1 if the jump occurs prior to time $t+jh$ or 0 otherwise. Also in equation (15) and in every equation from now onwards by S the following quantity is defined:

$$S = \frac{S}{1+\lambda t' E[k]}$$

Next natural logarithms are taken in (15) and their arithmetic average is considered, which is equivalent to the logarithm of the Geometric Average of prices. Furthermore, it is possible to divide by S since as it can be seen from (15) it is a common factor in all the equations. This arithmetic average can be written as:

$$\begin{aligned}
&\ln(GA/S) = ah \cdot \frac{1}{n} \cdot \frac{n(n+1)}{2} + \frac{\sigma}{n} \sum_{j=1}^{n} \sum_{k=1}^{j} Z_k^n + \frac{1}{n} \ln Y \sum_{j=1}^{n} I_{\tau<t+jh} \Rightarrow \\
&\ln(GA/S) = ah \cdot \frac{(n+1)}{2} + \frac{\sigma}{n} \sum_{k=1}^{n} Z_k^n (n-k+1) + \frac{1}{n} \ln Y \sum_{j=1}^{n} I_{\tau<t+jh}
\end{aligned} \tag{16}$$

where $a = r - d - \frac{1}{2}\sigma^2$

The first summation term in the RHS of equation (16) represents a sum of normally distributed random variables, which makes it itself a normal random variable. This variable will be called Z^n. The distribution of Z^n is normal and its mean and variance are estimated below[2]:

$$
\begin{aligned}
&Z^n \sim N(\mu, \sigma^2) \\
&\mu = a\left[T - \frac{h(n-1)}{2}\right] = aT_\mu \\
&\sigma = \sigma^2\left[T - \frac{h(n-1)(4n+1)}{6n}\right] = \sigma^2 T_\sigma
\end{aligned}
\qquad \textbf{(17)}
$$

The second summation term in the RHS of equation (16) is more difficult to analyse and here lies one of the main contribution of this paper.

Let us define a new variable M as follows:

$$M = \frac{1}{n}\sum_{j=1}^{n} I_{\tau < t + jh}$$

This random variable has the following properties:

$$
\begin{aligned}
&P(M = 0) = 1 - \theta \\
&P(M = 1/n) = \theta / n \\
&P(M = 2/n) = \theta / n \\
&\vdots \\
&P(M = 1) = \theta / n
\end{aligned}
\qquad \textbf{(18)}
$$

[2] It is assumed here that the averaging period will start immediately. If the averaging period has already started or if it will start in the future, these formulae will be slightly different.

At first glance, M doesn't seem to follow any standard distribution[3]. Intuitively, M will be equal to 1 if the jump occurs in the first subinterval, equal to $(1-1/n)$ if it occurs in the second one and so on. It will be equal to zero if there is no jump at all during the time period examined. Since the probability of a jump occurring in any particular subinterval is equal to θ/n the density function of M will look like (18).

Let W be defined as:

$$W = \ln(GA/S) \Rightarrow GA = Se^{W}$$

Using expression (13) for a call option we write[4]:

$$C = e^{-rT} E_t(\max[GA - K, 0]) = e^{-rT} E_t(\max[Se^{W} - K, 0]) = e^{-rT} E_t((Se^{W} - K)^{+})$$

This expectation is evaluated by conditioning upon M, expressing W as:

$W = Z + MY$ (Y is the lnY, variable for simplicity)

If M is known to be r/n, then[5]

$$W/M = Z + (r/n)Y$$

This shows that the conditional variable W/M being a sum of two normal variables is also normal. Formally:

$$W_r \equiv W/(M = r/n) \sim N(\mu_Z + \frac{r}{n}\mu_Y, \sigma_Z^2 + \left(\frac{r}{n}\right)^2 \sigma_Y^2) \equiv N(\mu_r, \sigma_r^2) \qquad \textbf{(19)}$$

[3] It can be proven that the continuous time limit of this distribution is a mixture of a Dirac function times $(1-\theta)$ plus θ times a uniform function in the interval [0,1].
[4] Only the case of call options will be dealt with since the corresponding put values can be easily calculated from the put call parity relation that holds for Asian options [See Levy (1992)].
[5] This r is not to be confused with the prevailing interest rate r.

$$C = e^{-rT}\sum_{r=0}^{n} E_t((Se^{W} - K)^{+} / M = r/n)P(M = r/n) =$$
$$e^{-rT}\{(1-\theta)E_t((Se^{W} - K)^{+} / M = 0) + \frac{\theta}{n}\sum_{r=1}^{n} E_t((Se^{W} - K)^{+} / M = r/n) \tag{20}$$

The first term in the summation above is the European Asian call option price conditional on no jumps occurring during the rest of the life of the option. The analytic formula for this term is given by the following equation:

$$C_{nj} = Se^{\mu+\frac{1}{2}\sigma^2}N(d_1) - KN(d_2)$$

$$d_1 = \frac{\ln(S/K) + \mu + \sigma^2}{\sigma},$$
$$d_2 = d_1 - \sigma$$

The second term is the term that takes into account the presence of the jump.

We will estimate the expectation inside the summation term.

$$E_t((Se^{W} - K)^{+} / M = r/n) = E_t((Se^{W_r} - K)^{+}) = E_t((Se^{W_r} - K)/W_r \geq \ln(K/S)) =$$
$$\int_{\ln(K/S)}^{\infty}(Se^{W_r} - K)f_{w_r}(w_r)dw_r = S\int_{\ln(K/S)}^{\infty} e^{W_r}e^{-\frac{(w_r-\mu_r)^2}{2\sigma_r^2}}\frac{1}{\sqrt{2\pi\sigma_r^2}}dw_r - K\int_{\ln(K/S)}^{\infty} e^{-\frac{(w_r-\mu_r)^2}{2\sigma_r^2}}\frac{1}{\sqrt{2\pi\sigma_r^2}}dw_r$$

After some calculations the value of the expectation is found to be:

$$E_t((Se^{W_r} - K)^{+}) = Se^{\mu_r+\frac{1}{2}\sigma_r^2}N(d_{1r}) - KN(d_{2r})$$

with

$$d_{1r} = \frac{\ln(S/K) + \mu_r + \sigma_r^2}{\sigma_r}, \quad r=0,1,\ldots,n$$
$$d_{2r} = d_{1r} - \sigma_r$$

and μ_r and σ_r are defined in (19).

Finally the value of a European Geometric Average Asian Call option for discrete sampling at n points is given by:

$$C = e^{-rT}\{(1-\theta)[Se^{\mu_z+\frac{1}{2}\sigma_z^2}N(d_{10}) - KN(d_{20})] + \frac{\theta}{n}\sum_{r=1}^{n}(Se^{\mu_r+\frac{1}{2}\sigma_r^2}N(d_{1r}) - KN(d_{2r}))\} \quad \textbf{(21)}$$

Geometric Asian Option Valuation With Continuous Sampling

Assuming that the sampling of the underlying asset's prices is being made continuously, then the limit as n tends to infinity of expression (21) is given by[6].

$$C = e^{-rT}\{(1-\theta)[Se^{\mu_z+\sigma_z^2}N(d_{10}) - KN(d_{20})] + \theta\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}(Se^{\mu_r+\sigma_r^2}N(d_{1r}) - KN(d_{2r}))\} \quad \textbf{(22)}$$

It can be proven[7] that

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) = \int_a^b f(x)dx \quad \textbf{(23)}$$

From (23) it is easily understood that formula (22) can now be written as:

$$C = e^{-rT}\{(1-\theta)[Se^{\mu_z+\sigma_z^2}N(d_{10}) - KN(d_{20})] + \theta\int_0^1 e^{(\mu_Z + x\mu_Y + \sigma_Z^2 + x^2\sigma_Y^2)}N(d_{1x})dx - \theta K\int_0^1 N(d_{2x})dx \quad \textbf{(24)}$$

,where x replaces r/n.

The integrals in the RHS of expression (24) can be computed using the Gauss method with 10 intermediate points. This method offers an approximation to the function of the same order as a 9th degree polynomial. Furthermore, since it only requires the estimation of the value of the function in only ten predefined points it is extremely fast. For the calculation of the cumulative normal distribution function an approximation that offers six-decimal-place accuracy[8] can be used.

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[6] In expression (19) n appears inside the mean and standard deviation of W_r.
[7] See Fischer E. "Intermediate Real Analysis",[1983], Springer-Verlag, NY,pg638, PROB 4.2.
[8] See M.Abramowitz and I.Stegun, *Handbook of Mathematical Functions,* [New York: Dover Publications, 1972].

6.5 Arithmetic Asian Option Valuation With Bernoulli Jump Diffusion Model

The valuation problem of Arithmetic Average Asian options (AA) is more complex than the corresponding Geometric one since it involves the estimation of the distribution of the sum of lognormal variables, which does not follow any known distribution.

Let τ^* be the time the jump occurs, and τ_i the time elapsed until the i^th time interval. The Arithmetic Average for an underlying asset that follows (14), written below for ease of reference, can be written as follows:

$$S(t`) = \frac{S}{1+\lambda E[k]t'} \exp\left[\left(r - d - \frac{1}{2}\sigma^2\right)t` + \sigma Z(t`)\right] Y(t') \tag{14}$$

$$AA = \frac{1}{n}\sum_{i=1}^{n} S\exp(a+\sigma Z_i) Y^{I_{\tau^*<\tau i}} \tag{25}$$

,where $I_{\tau^*<\tau i}$ is as in (14) the indicator function. Again as before, by S from equation (25) onwards the following quantity is referred to:

$$S = \frac{S}{1+\lambda t' E[k]} \tag{26}$$

From (25) if the event: {the jump occurs at the r-th interval} is named as J_r then the value of the Arithmetic Average Asian option can be expressed as:

$$\begin{aligned} E(AA-K)^{+} &= \sum_{r=1}^{n} E((AA-K)^{+}/J_r)P(J_r) = \\ &= (1-\theta)E\left(\frac{1}{n}\sum_{i=1}^{n} S\exp(a+\sigma Z_i) - K\right)^{+} \\ &+ \frac{\theta}{n}\sum_{r=1}^{n}\left[E\left(\frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y) - K\right)^{+}\right. \end{aligned} \tag{27}$$

For r being equal to zero the problem reduces to that of the valuation of an AA option assuming that the underlying asset follows a GBM but with a different initial stock price given from equation (26). For values of r other than zero the valuation problem is more complex since we have to evaluate two additional summation terms in each expectation. To evaluate each of these expectations we need to know the density function of each of these conditional distributions. The conditional distributions arise from the sum of k lognormal variables, for which, as it was mentioned earlier there is no closed form distributional expression.

There is, however, large body of evidence suggesting that the distribution of the sum of correlated lognormal variables is well approximated by another lognormal[9] distribution. Following Levy (1992) the variable equal to the sum is named $M(t)$, with unknown mean $a(t)$ and unknown variance $v(t)$ then these two parameters can be determined by considering the moment generating function for $X(t)=ln(M(t))$[10] given by:

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[9] See Fenton (1960), Janos (1970), Barakat (1976).

[10] We use the same notation as Levy (1992). This makes the comparison of the models used straightforward.

$$E[M(t)]^k = e^{ka(t)+0.5k^2v(t)^2} \quad \textbf{(28)}$$

More specifically, *M(t)* is defined as the sum of the stock prices conditional on the jump occurring at the *r*-th interval:

$$M(t) = \frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y)$$

In order to calculate these expectations for k=1 and k=2 a number of identities were used: The Z_j's and the lnY are normally distributed random variables. For normal random variables the following identity holds:

$$E(e^{tY}) = e^{t\mu+\frac{1}{2}t^2\sigma^2}\text{, where } Y \sim N(\mu,\sigma^2)$$

For example the following expectation is given by:

$$E(e^{a+\sigma Z_j}) = e^a e^{0+\frac{1}{2}\sigma^2 jdt} = e^{a+\frac{1}{2}\sigma^2 jdt} \quad \textbf{(29)}$$

,where $dt=T/n$ is the length of one time interval and T is the time to maturity.

The summation of n terms of the form of (29), with respect to j is a sum of n terms of a geometric progress. Therefore it is:

$$\sum_{j=1}^{r-1} e^{a+\frac{1}{2}\sigma^2 jdt} = e^a \frac{e^{\frac{1}{2}\sigma^2 Tr/n} - e^{\frac{1}{2}\sigma^2 T/n}}{e^{\frac{1}{2}\sigma^2 T/n} - 1}$$

Furthermore it is:

$$\sum_{j=}^{r} e^{a+\frac{1}{2}\sigma^2 jdt} = e^a \frac{e^{\frac{1}{2}\sigma^2 T(r+1)/n} - e^{\frac{1}{2}\sigma^2 T/n}}{e^{\frac{1}{2}\sigma^2 T/n} - 1}$$

When dealing with powers of sums that are greater than 1 then we have to estimate the expectation of products of sums of the form:

$$\sum_{i=1}^{r-1}\sum_{j=r}^{n}\exp(\sigma Z_i+\sigma Z_j)$$

The variables Z_i and Z_j are correlated variables so (29) cannot be applied directly. Instead the double summation is expressed in the following way:

$$\sum_{i=1}^{r-1}\sum_{j=r}^{n}\exp(\sigma Z_i+\sigma Z_j)=\sum_{i=1}^{r-1}\sum_{j=r}^{n}\exp(\sigma Z_i+\sigma Z_i+\sigma(Z_j-Z_i)) \tag{30}$$

Here Z_i and Z_j-Z_i are independent since a Brownian motion has independent increments. Therefore (29) applies to the product of the two exponentials in (30) and the calculation of the summation is now feasible.

In order to calculate the first and second moments of $M(t)$ the following quantities are defined for notational simplicity:

$$\frac{e^{(k\sigma'+\lambda\alpha)(n+1)}-e^{(k\sigma'+\lambda\alpha)r}}{e^{k\sigma'+\lambda\alpha}-1}=k_{\lambda,n}$$

$$\frac{e^{(k\sigma'+\lambda\alpha)(r+1)}-e^{(k\sigma'+\lambda\alpha)}}{e^{k\sigma'+\lambda\alpha}-1}=k_{\lambda,r}$$

$$\frac{1}{e^{k\sigma'+\lambda\alpha}-1}=k_{\lambda}$$

$$\mu_{Y,k}=e^{k\mu_Y+\frac{1}{2}\sigma_Y^2k^2} \qquad \sigma'=\tfrac{1}{2}\sigma^2 \qquad \alpha=r-d-\tfrac{1}{2}\sigma^2$$

Then, the first and second moments, $E[M(t)]$ and $E[M^2(t)]$ are given by:

$$E[M(t)]=\frac{S}{n}(1_{1,r}+\mu_{Y,1}1_{1,n}) \tag{31}$$

$$E[M^2(t)] = \frac{S^2}{n^2}\{4_{2,r} + 2\cdot 3_1(4_{2,r} - e^{3\sigma'+a}1_{1,r}) + \mu_{Y,2}[4_{2,n} + 2\cdot 3_1(4_{2,n} - e^{(3\sigma'+a)r}1_{1,n})] + 2\mu_{Y,1}3_{1,r}1_{1,n}\} \quad \textbf{(32)}$$

It is straightforward to arrive to these formulae since one is essentially summing the first n terms of a geometric progression.

From (28) it is obvious that the first two moments of *M(t)* are complete and sufficient statistics to determine *a(t)* and *v(t)*.

Solving for *a(t)* and *v(t)* we obtain:

$$\begin{aligned} a(t) &= 2\ln E[M(t)] - \tfrac{1}{2}\ln E[M(t)^2] \\ v(t) &= \sqrt{\ln E[M(t)^2] - 2\ln E[M(t)]} \end{aligned} \quad \textbf{(33)}$$

For each value of r a different value for *a(t)* and *v(t)* is calculated. The first and second moments of *M(t)* can be estimated in closed form, being a summation of exponential terms. Equipped with the distribution of each term in the summation in (27), the expectations can be easily found to be equal to:

$$E\left(\frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y) - K\right)^+ = e^{a(t)_r + \frac{1}{2}v(t)_r^2} N(d_{1r}) - KN(d_{2r}) \quad \textbf{(34)}$$

with

$$\begin{aligned} d_{1r} &= \frac{-\ln(K) + a(t)_r + v(t)_r^2}{v(t)_r}, \\ d_{2r} &= d_{1r} - v(t)_r \end{aligned}$$

Finally by substituting back the above expression in (27) and discounting, the value of the European call option is estimated. In order to calculate option prices for the corresponding European Asian puts it suffices to invoke the put-call parity relationship that holds for this type of contracts.

Put-cal parity relationship has been proven to hold when the underlying asset is an exchange rate[11]and it is proved as follows[12]:

Defining as r_d and r_f the domestic and foreign interest rates respectively and assuming frictionless markets a long position in a call option and a short position in a put option with the same strike K is considered. This is equivalent to a position that pays

$$\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K$$

units of domestic currency at maturity $t_n=T$. the value of this position at inception depends on the spot prices $S(t_i)$ which can be hedged in the forward market. For each time t_i arrange at time t to sell forward $\frac{1}{n}e^{-r_d(T-t_i)}$ units of foreign currency for delivery at time t_i. The value of such a hedge is zero at time t. Subsequently at each t_i these forward positions are closed a the prevailing spot rates and a net value of:

$$\frac{1}{n}e^{-r_d(T-t_i)}(F_{t,t_i} - S(t_i))$$

is obtained. In the expression above F_{t,t_i} is the forward value of the exchange rate calculated at time t for delivery at time t_i. These flows may be placed on deposit (or

[11] See Levy (1992), Turnbull and Wakeman (1991), Kemna and Vorst (1991)
[12] The analysis that follows relies heavily in the analysis presented in Levy (1992) with some simplifications.

borrowed) for (T-t_i) realising a total value of:

$$\frac{1}{n}\sum_{i=1}^{n}[F_{t,t_i} - S(t_i)]$$

Finally setting off this amount against the maturity value of the call and put position, we have a known domestic currency amount, which can be discounted at the risk free rate to give:

$$V(t) = e^{-r_d(T-t)}\left[\frac{S(t)}{n}\sum_{i=1}^{n} e^{(r_d-r_f)(t_i-t)} - K\right] \Rightarrow$$

$$V(t) = e^{-r_d(T-t)}\left[\frac{S(t)}{n}\frac{e^{gh} - e^{g(n+1)h}}{1-e^{gh}} - K\right]$$

where, $g=r_d-r_f$, and t_i-t was replaced by $i*h$. it follows that if $C(t)$ and $P(t)$ denote the values for the call and put options respectively then to avoid arbitrage it should be:

$$C(t)-P(t)=V(t) \tag{35}$$

Expression (35) above represents the put call parity relationship for Arithmetic Asian option on currencies.

6.6 An alternative specification for the distribution of the sum of lognormal variables

The approach that is followed in this study closely resembles the method presented in Levy (1992). It is a generalization of the latter method within a more realistic framework. Turnbull and Wakeman (1991) provided closed form solutions to the Arithmetic Asian valuation problem in a more general setting than that of Levy (1992). Assuming again lognormality for the terminal risk-neutral distribution of the underlying asset, they approximated the unknown distribution of the sum of lognormal variables with an Edgeworth series expansion around a known lognormal distribution.

As it was previously mentioned Edgeworth series expansions were first used in vanilla option pricing by Jarrow and Rudd (1982) and again after 15 years an improved version, was used by Corrado and Su (1997). An Edgeworth series expansion is the probabilistic analog of a Taylor series expansion. The distribution function of a variable is approximated with the help of its cumulants. As in Taylor series expansion the function is approximated around a specific point x_0, here the distribution function of a random variable is approximated around an *a priori* given approximating distribution. So if the true probability function is denoted by $f(y)$ and the approximating one by $a(y)$ then:

$$f(y) = a(y) + \frac{c_2}{2!}\frac{d^2 a(y)}{dy^2} - \frac{c_3}{3!}\frac{d^3 a(y)}{dy^3} + \frac{c_4}{4!}\frac{d^4 a(y)}{dy^4} + e(y) \qquad \textbf{(36)}$$

,where $c_2 = \chi_2(F) - \chi_2(A); c_3 = \chi_3(F) - \chi_3(A); c_4 = \chi_4(F) - \chi_4(A) + 3c_2^2;$

and $\chi_j(F)$ [$\chi_j(A)$] is the j^{th} cumulant of the exact [approximating] distribution, and $e(y)$ is residual error term. If a random variable Y has a cumulative distribution F, the first four cumulants are:

$\chi_1(F)=E(Y)$ $\chi_2(F)=E(Y-E(Y))^3$

$\chi_3(F)=E(Y-E(Y))^3$ $\chi_4(F)=E(Y-E(Y))^4-3E(Y-E(Y))^3$

where all expectations are with respect to the distribution F. From the above equations it is easily understood that in order to be able to evaluate an Arithmetic Asian option the quantities $E(Y^m)$ have to be calculated for m=1,2,3,4.

The call option value for a variable that is described as an Edgeworth series expansion around a known distribution is then given by the formula below:

$$C(Y;f) = e^{-r(T-t)} \left[\begin{array}{l} C(Y,a) \\ \quad - \dfrac{(\kappa_3(f)-\kappa_3(a))}{3!}\dfrac{dC(K)}{dY} + \dfrac{(\kappa_4(f)-\kappa_4(a))}{4!}\dfrac{d^2C(K)}{dY^2} \end{array} \right] \tag{37}$$

In equation (37) *C(Y,a)* denotes the value of the call option under the approximating distribution and also the first and second derivatives of this function with respect to the underlying evaluated at the strike price are also present.

The *Y* variable in the previous expression is the variable *M(t)* first found in equation (28):

$$E(Y^m) = E\left(\frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y)\right)^m, m = 1,2,3,4. \tag{38}$$

the first two expectations corresponding to m=1,2, have already been computed in (31) and (32).

Within the Edgeworth series expansion framework the expectation of the third and fourth moments of the *M(t)* variable have to be estimated. Proceeding in a way similar to the one outlined in the above discussion, closed form expressions can be derived for the cumulants of each conditional distribution that will include r and n. From these expressions the coefficients of the Edgeworth series expansion are calculated and finally the expectation in (38) can be calculated. Finally to obtain the Asian option price we sum the expectations from 1 to n with respect to r.

The Edgeworth series expansion method requires an *a priori* choice of an approximating distribution. The latter can be selected in a number of ways. Following available evidence in the literature, (see Footnote 9) the probability that the distribution of the sum of lognormal variables can be well approximated by another lognormal variable. This evidence suggests that a good choice for the approximating distribution would be one that from the family of lognormal distributions. Turnbull and Wakeman (1992) do not make clear how they make their choice. Within the Bernoulli jump diffusion framework two different ways can be proposed for the parameterisation of the approximating distribution.

The first way is a combination of the methods described in Turnbull and Wakeman (1991) and Levy (1992). Every term in (34) is expanded according to an Edgeworth series expansion around its own approximating distribution. The latter will be chosen as in Levy (1992) so that the first two moments of the true and the approximating distribution be equal. This method is expected to present the best results in terms of accuracy but it will have the greater computational cost. However, the fact that the first two moments of the true and the approximating distributions are equal will somewhat simplify the calculations.

The second method is to choose one approximating distribution for all Edgeworth series expansions of the terms in (34). This way the computational cost will be less but at the expense perhaps of additional accuracy. However, the methodology that is followed in this study does produce satisfactory results in terms of pricing accuracy that justify the avoidance of the additional complexity that an Edgeworth series would introduce into the model.

6.6.1 Pricing when the averaging period has already started

When the averaging period has already started then the first, say m terms, of the summation in (27) are known. This has as an effect[13] the reduction of the variance of the sum, giving rise to smaller option prices. If we denote with *Bf(t)* the sum of all the values up to time *t* and with *Af(t)* the sum of the values from time *t* until maturity, then:

implying that

$$M(t) > K \Rightarrow Af(t)/n > K - Bf(t)/n \tag{39}$$

From (39) it is obvious that the distribution of *Af(t)* has to be estimated in order to calculate the expectation terms in (33). The moments of the distribution function of *Af(t)* are calculated and the options are evaluated in the new effective strike price *K-Bf(t)/n*.

6.6.2 Pricing when the averaging period will start in the future

The case when the averaging period is forward starting is almost the same with the case when we are standing exactly in the beginning of it. The only difference that has to be noticed is that different cases should be considered depending on the event of the jump occurring before or during the averaging period. This implies that the price of the option will include the sum of three different terms and not just two as in (27). More specifically, if we denote with A_{bf}, A_{af} and A_{nj} the value of the Asian option conditional on the jump occurring before or after the averaging period or not occurring at all respectively, then the value of the Asian option is given by:

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[13] See also Levy (1992), Turnbull and Wakeman (1991).

$$AO = \theta A_{nj} + (1-\theta\frac{m}{N})A_{bf} + (1-\theta\frac{n}{N})A_{af}$$

where it was assumed that there were m time periods preceding the start of the averaging period, and n time periods inside the averaging period ($N=m+n$).

6.7 Simulation Results

In order to determine the validity of the theoretically produced results, a Monte Carlo simulation experiment was conducted so as to check whether the option prices computed by the Bernoulli jump diffusion model, are in accordance with the ones postulated by the assumed distribution. The Monte Carlo experiment was designed in such a way so as to be consistent with the stochastic process (3):

$$\frac{dS}{S} = (r - d - \lambda E(k))dt + \sigma_w dW_t + kdq \qquad (3)$$

For each Asian call option an experiment of 10,000 replications was carried with S_t=1.5, r=0.15, d=0.1, T=1, σ_j=0.005 and θ=0.6. (S_t is the stock price today, r is the interest rate, d the dividend yield, T the time to maturity, σ_j is the volatility of the jump and θ is the Bernoulli distribution parameter).

As in Kemna and Vorst (1990), the Geometric Average option was used as a control variate for the Monte Carlo simulation. The value for the Geometric option was calculated from expression (21) with the appropriate amendments depending on the starting time of the averaging period. The results presented in Tables III, IV and V, are for Asian options with daily fixing for three different possible values for the mean of the jump.

In almost all cases, theoretical option prices using the extended Levy (1992), method have less than one percent deviation from their corresponding Monte Carlo estimates. When the option is in the money, the accuracy of the method does reach even higher levels. The method seems to be performing comparatively not as efficiently, when the case of out of the money options with high volatility of the underlying is examined, the averaging period starting now and with negative mean jump size. In the worse case the percentage deviation from the Monte Carlo estimated price is around two percent implying that the method continues to remain adequately efficient.

Comparing the values of otherwise identical options for different averaging period starting times, it is observed that options are more expensive when the averaging period will start in the future and less expensive when it has already started. This is an expected scenario since in the latter case some of the observations that are being calculated in the averaging of the option are already known, thus reducing the variance of the total sum. Furthermore, option price estimates are more expensive for greater values of the volatility of the underlying and for positive mean jump size, compared to prices for lower volatility and negative or zero jump size as it was intuitively expected.

6.7.1 Alternative Sampling Frequencies

In the analysis presented so far a daily frequency was assumed for the sampling of underlying asset's prices. In this section we perform further simulation tests of the Bernoulli jump diffusion model using alternative sampling frequencies. Option prices estimated by the valuation model were compared to those generated by Monte Carlo simulation for weekly, monthly and quarterly sampling frequencies. The results are exhibited in Tables VI, VII and VIII. They concern only the case when the averaging period will start at the next time interval. This choice was made because it was evident from previous simulation results that in this case the most sizeable pricing inconsistencies were present.

Looking at the tabulated results, it is clear that the proposed method provides numerical values that are consistent with those of the Monte Carlo simulation. The deviations from the theoretical price of the options are adequately small for every combination of standard deviation, mean jump size and moneyness. The largest deviation from theoretical prices is observed for options that are deep in the money and have high volatility. Even in that case though, the deviations remain well under one per cent, proving that the proposed methodology is indeed very accurate.

It can also be noticed that, otherwise identical options become more expensive as the observation frequency decreases from daily to quarterly. This is to be expected, since the fewer observation points there are during the life of the option, the more the Asian option approximates its European vanilla counterpart, hence being more expensive.

In a word, irrespective of the different values of the parameters used in the estimation procedure the model provides theoretically consistent estimates.

Table III. Simulation results for the case that the averaging period starts at the next time period

Daily Averaging		MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T=t+dt		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.0040728	0.00395	0.00328	0.003083	0.00274	0.002522
	S/K=0.95	0.0197639	0.019586	0.01699	0.016687	0.01483	0.0144557
S=0.1	S/K=1	0.0548518	0.05475	0.04981	0.049554	0.04534	0.0449925
	S/K=1.05	0.1042140	0.104139	0.09812	0.097911	0.09224	0.0919681
	S/K=1.1	0.1569415	0.156817	0.15092	0.150705	0.14492	0.1446485
Standard Error	6.86E-07						
	S/K=0.9	0.0258688	0.025433	0.02402	0.023358	0.02234	0.0215859
	S/K=0.95	0.0497718	0.049521	0.04683	0.04632	0.04415	0.0434732
S=0.2	S/K=1	0.0827397	0.082753	0.07874	0.078555	0.07508	0.0746903
	S/K=1.05	0.1226237	0.122847	0.11809	0.117988	0.11363	0.1133924
	S/K=1.1	0.1664334	0.166658	0.16148	0.161529	0.15668	0.1565731
Standard Error	3.41E-05						
	S/K=0.9	0.0537454	0.053304	0.05155	0.050814	0.04965	0.0485499
	S/K=0.95	0.0803558	0.080215	0.07762	0.077072	0.07482	0.0741551
S=0.3	S/K=1	0.1117960	0.112262	0.10867	0.108568	0.10551	0.1050867
	S/K=1.05	0.1474358	0.148094	0.14377	0.144021	0.14007	0.1401399
	S/K=1.1	0.1854705	0.186241	0.18147	0.181970	0.17767	0.1778527
Standard Error	6.79E-05						

Table IV. Simulation results for the case that the averaging period will start in the future.

6.7.1.1.1.1	Daily Averaging	MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T= t-t0 (=t-0.5)		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	S/K=0.95	0.00083	0.00091	0.00068	0.00060	0.00052	0.00045
S=0.1	S/K=1	0.02989	0.03001	0.02653	0.02656	0.02365	0.02352
	S/K=1.05	0.09318	0.09321	0.08947	0.08946	0.08571	0.08573
	S/K=1.1	0.15305	0.15309	0.14969	0.14970	0.14636	0.14633
Standard Error	2.05E-05						
	S/K=0.9	0.00050	0.00055	0.00045	0.00044	0.00032	0.00036
	S/K=0.95	0.00773	0.00784	0.00693	0.00683	0.00621	0.00603
S=0.2	S/K=1	0.03880	0.03906	0.03613	0.03624	0.03377	0.03369
	S/K=1.05	0.09402	0.09417	0.09052	0.09058	0.08701	0.08709
	S/K=1.1	0.15296	0.15311	0.14969	0.14972	0.14640	0.14637
Standard Error	4.53E-05						
	S/K=0.9	0.00423	0.00407	0.00385	0.00363	0.00352	0.00327
	S/K=0.95	0.01767	0.01775	0.01642	0.01638	0.01548	0.01519
S=0.3	S/K=1	0.04924	0.04960	0.04681	0.04705	0.04458	0.04468
	S/K=1.05	0.09797	0.09834	0.09477	0.09508	0.09182	0.09193
	S/K=1.1	0.15362	0.15380	0.15040	0.15049	0.14712	0.14723
Standard Error	6.50E-05						

Table V. Simulation results for the case that the averaging period has already started.

Daily Averaging		MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T=t+t0 (=t+0.5)		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.0228353	0.022854	0.028472	0.028513	0.01759	0.0176541
	S/K=0.95	0.0501526	0.050118	0.064888	0.064813	0.04133	0.0413077
S=0.1	S/K=1	0.0884564	0.088438	0.117633	0.117605	0.07673	0.0767289
	S/K=1.05	0.1332360	0.133224	0.180789	0.180785	0.12015	0.120158
	S/K=1.1	0.1796821	0.179668	0.247358	0.247326	0.16667	0.1666786
Standard Error	2.22E-05						
	S/K=0.9	0.0663033	0.066436	0.089975	0.090215	0.06000	0.0601765
	S/K=0.95	0.0951888	0.095322	0.130368	0.130511	0.08751	0.087711
S=0.2	S/K=1	0.1282655	0.128382	0.176783	0.177042	0.11954	0.1197518
	S/K=1.05	0.1640809	0.164189	0.227596	0.227796	0.15478	0.1549517
	S/K=1.1	0.2013292	0.201406	0.280725	0.280868	0.19190	0.1919802
Standard Error	3.01E-05						
	S/K=0.9	0.1111724	0.111909	0.154526	0.15558	0.10546	0.1060956
	S/K=0.95	0.1398065	0.140541	0.194933	0.196076	0.13338	0.1341584
S=0.3	S/K=1	0.1701691	0.171028	0.238416	0.239375	0.16355	0.1642656
	S/K=1.05	0.2019912	0.202629	0.283356	0.284407	0.19500	0.1957005
	S/K=1.1	0.2341059	0.234757	0.329392	0.330331	0.22713	0.2278439
Standard Error	6.22E-05						

Table VI. Simulation results for a weekly averaging frequency

Weekly Averaging		MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T=t		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.004222	0.004227	0.00339	0.003308	0.00284	0.00270923
	S/K=0.95	0.020175	0.020261	0.01733	0.017279	0.01513	0.0149746
S=0.1	S/K=1	0.055423	0.055636	0.05027	0.050354	0.04574	0.04571145
	S/K=1.05	0.104762	0.104988	0.09855	0.098663	0.09260	0.09263044
	S/K=1.1	0.157448	0.1576	0.15131	0.151378	0.14521	0.14521419
Standard Error	9.11E-06						
	S/K=0.9	0.026435	0.026338	0.02452	0.0242	0.02284	0.02237072
	S/K=0.95	0.050432	0.050672	0.04736	0.047404	0.04472	0.04449298
S=0.2	S/K=1	0.083514	0.083997	0.07945	0.079729	0.07573	0.07579869
	S/K=1.05	0.123331	0.124047	0.11866	0.119113	0.11422	0.11444688
	S/K=1.1	0.167069	0.167742	0.16203	0.162531	0.15719	0.15749602
Standard Error	2.74E-05						
	S/K=0.9	0.05476	0.054736	0.05252	0.052187	0.05049	0.04986639
	S/K=0.95	0.081207	0.081801	0.07835	0.078598	0.07578	0.0756229
S=0.3	S/K=1	0.112995	0.113894	0.10949	0.110137	0.10635	0.10659667
	S/K=1.05	0.148496	0.149678	0.14458	0.14554	0.14088	0.14159624
	S/K=1.1	0.186302	0.187717	0.1823	0.183376	0.17835	0.17919162
Standard Error	6.43E-05						

Table VII. Simulation results for a monthly averaging frequency

Monthly Averaging		MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T=t		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.005278	0.005283	0.004249	0.004164	0.00355	0.00341605
	S/K=0.95	0.022591	0.022681	0.019414	0.019374	0.01691	0.01679016
S=0.1	S/K=1	0.058534	0.058741	0.053026	0.053101	0.04816	0.04812835
	S/K=1.05	0.10773	0.10795	0.101127	0.101216	0.09478	0.09480182
	S/K=1.1	0.160199	0.160325	0.153588	0.153637	0.14701	0.14702267
Standard Error	8.38E-06						
	S/K=0.9	0.029683	0.029604	0.027437	0.027219	0.02554	0.0251654
	S/K=0.95	0.054568	0.054753	0.051192	0.051213	0.04822	0.04804466
S=0.2	S/K=1	0.087837	0.088375	0.083571	0.083814	0.07951	0.0796065
	S/K=1.05	0.127675	0.128261	0.122601	0.123009	0.11784	0.11803954
	S/K=1.1	0.17095	0.171559	0.165526	0.165991	0.16030	0.16061607
Standard Error	2.92E-05						
	S/K=0.9	0.059787	0.05984	0.05722	0.057054	0.05487	0.05450871
	S/K=0.95	0.086656	0.087407	0.083746	0.083957	0.08078	0.08074249
S=0.3	S/K=1	0.118713	0.11964	0.11503	0.115619	0.11151	0.11182576
	S/K=1.05	0.154064	0.155253	0.14989	0.150834	0.14596	0.14661988
	S/K=1.1	0.19172	0.192922	0.187136	0.188275	0.18289	0.1837963
Standard Error	6.50E-05						

Table VIII. Simulation results for a quarterly averaging frequency.

Quarterly Averaging		MC	Levy ext.	MC	Levy ext.	MC	Levy ext.
T=t		my=0.02		my=0		my=-0.02	
	S/K=0.9	0.0087	0.0087	0.0069	0.0069	0.0057	0.0056
	S/K=0.95	0.0295	0.0296	0.0251	0.0251	0.0216	0.0215
S=0.1	S/K=1	0.0675	0.0676	0.0602	0.0602	0.0538	0.0538
	S/K=1.05	0.1170	0.1171	0.1078	0.1079	0.0993	0.0993
	S/K=1.1	0.1694	0.1695	0.1596	0.1596	0.1500	0.1500
Standard Error	7.37E-06						
	S/K=0.9	0.0390	0.0390	0.0356	0.0354	0.0326	0.0324
	S/K=0.95	0.0661	0.0663	0.0611	0.0612	0.0568	0.0567
S=0.2	S/K=1	0.1006	0.1009	0.0942	0.0944	0.0883	0.0884
	S/K=1.05	0.1405	0.1409	0.1328	0.1332	0.1257	0.1259
	S/K=1.1	0.1834	0.1838	0.1748	0.1751	0.1666	0.1669
Standard Error	2.93E-05						
	S/K=0.9	0.0741	0.0743	0.0699	0.0700	0.0663	0.0660
	S/K=0.95	0.1027	0.1033	0.0977	0.0979	0.0929	0.0929
S=0.3	S/K=1	0.1353	0.1362	0.1294	0.1298	0.1235	0.1239
	S/K=1.05	0.1709	0.1718	0.1639	0.1646	0.1573	0.1578
	S/K=1.1	0.2081	0.2090	0.2003	0.2011	0.1929	0.1935
Standard Error	6.81E-05						

6.8 ASIAN GREEKS

Expressions for the two most widely used hedging parameters "Greeks" of Asian options, the Delta and the Gamma are provided in this section. The derivation of the formulae has been made through differentiation of the pricing equations for the case that the averaging period starts immediately. It is trivial to extend the Greeks' derivation for the other two cases examined in this study.

6.8.1 Geometric Average Option

The value of a European Geometric Average Asian Call option for discrete sampling at *n* points is given by formula (21) written below for ease of reference:

$$C = e^{-rT}\{(1-\theta)[Se^{\mu_z+\frac{1}{2}\sigma_z^2}N(d_{10}) - KN(d_{20})] + \frac{\theta}{n}\sum_{r=1}^{n}(Se^{\mu_r+\frac{1}{2}\sigma_r^2}N(d_{1r}) - KN(d_{2r}))\} \quad (21)$$

with

$$d_{1r} = \frac{\ln(S/K) + \mu_r + \sigma_r^2}{\sigma_r}, \quad r=0,1,\ldots,n$$

$$d_{2r} = d_{1r} - \sigma_r$$

,where

$$\mu_r = \mu_Z + \frac{r}{n}\mu_Y,$$

$$\sigma_r^2 = \sigma_Z^2 + \left(\frac{r}{n}\right)^2 \sigma_Y^2$$

and μ_Z and μ_Y as in (19)

For the purpose of differentiation it is best to replace the mean μ_r with the following quantity:

$$\mu_r = \mu_r - \ln(1 + \lambda TE[k])$$

This will enable an easier interpretation of the results. The quantity S here represents simply the spot price of the underlying asset.

6.8.1.1 Delta

It is well known that the delta of a European vanilla call option with payoff according to the Black and Scholes assumptions is given by the following formula:

$$Delta_{BS} = e^{-dt} N(d_1) \qquad \textbf{(40)}$$

with

$$d_1 = \frac{\ln(S/K) + (r - d + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

The delta of a Geometric Average option priced according to (34) a 2-component mixtures lognormal distribution is given by the following formula:

$$Delta_{GEOM} = \frac{e^{-dt}}{1 + \lambda E[k]T}[(1-\theta)N(d_{10}) + \frac{\theta}{n}\sum_{r=1}^{n} N(d_{1r})] \qquad \textbf{(41)}$$

with d_{10} and d_{1r} from above.

6.8.1.2 Gamma

The Gamma of an option is simply the first derivative of the Delta. Thus by differentiating (34) once more an expression for the value of Gamma for a Geometric Average Asian Option is derived. The Black and Scholes Gamma of a European vanilla option is given by the following formula:

$$Gamma_{BS} = e^{-dt}\frac{N'(d_1)}{S\sigma\sqrt{T}} \tag{42}$$

Applying the same reasoning as before an expression for the Gamma in question can be obtained:

$$Gamma_{GEOM} = \frac{e^{-dt}}{1+\lambda E[k]T}[(1-\theta)\frac{N'(d_{10})}{S\sigma\sqrt{T}} + \frac{\theta}{n}\frac{1}{S}\sum_{r=1}^{n}\frac{N'(d_{1r})}{S\sigma_r\sqrt{T}}] \tag{43}$$

6.8.2 Arithmetic Average Option

As it was proven earlier, the Arithmetic Average Asian option can be expressed as:

$$C_{ARITHM} = (1-\theta)E\left(\frac{1}{n}\sum_{i=1}^{n} S\exp(a+\sigma Z_i) - K\right)^+$$
$$+\frac{\theta}{n}\sum_{r=1}^{n}\left[E\left(\frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y) - K\right)^+\right]$$

Each term will be considered separately:

$$E\left(\frac{1}{n}\sum_{j=1}^{r-1} S\exp(a+\sigma Z_j) + \frac{1}{n}\sum_{j=r}^{n} S\exp(a+\sigma Z_j + \ln Y) - K\right)^+ = e^{a(t)_r + \frac{1}{2}v(t)_r^2}N(d_{1r}) - KN(d_{2r})$$

where

$$a(t) = 2\ln E[M(t)] - \tfrac{1}{2}\ln E[M(t)^2]$$
$$v(t) = \sqrt{\ln E[M(t)^2] - 2\ln E[M(t)]}$$

and

$$d_{1r} = \frac{-\ln(K) + a(t)_r + v(t)_r^2}{v(t)_r},$$
$$d_{2r} = d_{1r} - v(t)_r$$

Again for the purposes of differentiation the following substitution will be made:

$a(t)_r = a(t)_r - \ln(1+\lambda E[k]T)$

6.8.2.1 Delta

If the right hand side of equation (27) is named C_r then the "delta" of each term (Delta_r) is given by the following equation:

$$Delta_r = e^{a(t)_r + \frac{1}{2}v(t)_r^2} N(d_{1r})\left(a'(t) + v'(t)v(t)\right) + e^{a(t)_r + \frac{1}{2}v(t)_r^2} N'(d_{1r})d'_{1r} - KN'(d_{2r})d'_{2r} \qquad \textbf{(44)}$$

where

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2}$$

and the rest of the primes represent partial derivative with respect to S.

The derivatives present in equation (44) are given below:

$$\frac{\partial d_{1r}}{\partial S} = \frac{\frac{\partial a(t)}{\partial S} v(t) - a(t)\frac{\partial v(t)}{\partial S}}{v^2(t)} + \frac{\partial v(t)}{\partial S}$$

$$\frac{\partial d_{2r}}{\partial S} = \frac{\partial d_{1r}}{\partial S} - \frac{\partial v(t)}{\partial S}$$

,furthermore:

$$\frac{\partial a(t)}{\partial S} = 2\frac{1}{E[M(t)]}\frac{\partial E[M(t)]}{\partial S} - \frac{1}{2}\frac{1}{E[M^2(t)]}\frac{\partial E[M^2(t)]}{\partial S}$$

$$\frac{\partial v(t)}{\partial S} = \frac{1}{2\sqrt{\ln E[M^2(t)] - 2\ln E[M(t)]}}\left(2\frac{1}{E[M^2(t)]}\frac{\partial E[M^2(t)]}{\partial S} - \frac{1}{2}\frac{1}{E[M(t)]}\frac{\partial E[M(t)]}{\partial S}\right)$$

and finally:

$$\frac{\partial E[M(t)]}{\partial S} = \frac{1}{S} E[M(t)]$$
$$\frac{\partial E[M^2(t)]}{\partial S} = \frac{2}{S} E[M^2(t)]$$

Knowing each individual Delta_r it is possible to calculate the delta of the Arithmetic Average Option, which will simply be the weighted average of the individual "deltas".

$$Delta_{ARITHM} = (1-\theta)Delta_0 + \frac{\theta}{n}\sum_{r=1}^{n} Delta_r \tag{45}$$

6.8.2.2 Gamma

By differentiating once more equation (41) we get an expression for the Gamma_r, or in other words, the "Gamma" of each term in (27).

$$\begin{aligned} Gamma_r = e^{a(t)_r + \frac{1}{2}v(t)_r^2} (N(d_{1r})(a' + v'v)^2 + N(d_{1r})(a'' + v''v + v'^2) + 2(a' + v'v)N'(d_{1r})d'_{1r} + \\ + N'(d_{1r})d''_{1r} + N''(d_{1r})d'^2_{1r} - KN''(d_{2r})d'^2_{2r} - KN'(d_{2r})d''_{2r} \end{aligned} \tag{46}$$

The derivatives present in equation (46) are given below for completeness:

$$\frac{\partial^2 d_{1r}}{\partial S^2} = \frac{(a''v - av'')v - 2v'(a'v - av')}{v^3(t)} + v''$$

$$\frac{\partial^2 d_{2r}}{\partial S^2} = \frac{\partial^2 d_{1r}}{\partial S^2} - v''$$

For $a(t)$ and $v(t)$ it is:

$$\frac{\partial^2 a(t)}{\partial S^2} = -2\frac{1}{E^2 M} E'^2 M + 2\frac{1}{EM} E'' M + \frac{1}{2}\frac{1}{E^2 M^2} E'^2 M^2 - \frac{1}{2}\frac{1}{EM^2} E'' M^2$$
$$\begin{aligned} \frac{\partial^2 v(t)}{\partial S^2} = \frac{1}{4\sqrt[3]{\ln EM^2 - 2\ln EM}} \left(2\frac{E'M^2}{EM^2} - \frac{1}{2}\frac{E'M}{EM}\right)^2 + \\ + \frac{1}{2\sqrt{\ln EM^2 - 2\ln EM}} \left(-2\frac{E'^2 M^2}{E^2 M^2} + 2\frac{E''M^2}{EM^2} + \frac{1}{2}\frac{E'^2 M}{E^2 M} - \frac{1}{2}\frac{E''M}{EM}\right) \end{aligned}$$

Finally for the two moments the second derivatives are given from the two equations below:

$$\frac{\partial^2 E[M(t)]}{\partial S^2} = \frac{-1}{S^2} E[M(t)] + \frac{1}{S} E'[M(t)] = 0$$

$$\frac{\partial E^2[M^2(t)]}{\partial S^2} = \frac{-2}{S^2} E[M^2(t)] + \frac{2}{S} E'[M^2(t)] = \frac{2}{S^2} E[M^2(t)]$$

Again, due to the additivity property of the derivative operator, in order to calculate the Gamma of the Arithmetic Average Asian Option, we obtain an expression similar to (43).

$$Gamma_{ARITHM} = (1-\theta)Gamma_0 + \frac{\theta}{n}\sum_{r=1}^{n} Gamma_r \tag{47}$$

6.9 CONCLUSIONS

Closed form expression for Geometric and Arithmetic call and put options are provided in this chapter, along with expressions for the risk management parameters. The results have been generated under the assumption that the sum of lognormal variables is also a lognormal variable. An analysis has been performed for different averaging frequencies and with different hypothesis concerning the relative position of the valuation time and the start of the averaging period. Monte Carlo simulation experiments validate the theoretical results.

7 CHAPTER SEVEN: BASKET OPTIONS

Basket options are exotic derivative instruments that are very frequently traded among practitioners in the Over The Counter (OTC) market. One of the main merits basket options have is that a basket option is cheaper than the basket of single options on each of the individual stocks that comprise the basket. This is due to a very well known result from portfolio theory, namely that the volatility of the basket of stocks is decreased provided that the individual components are not perfectly correlated (diversification effect). Fund managers bearing exposure on a portfolio of stocks, hedge that risk by the purchase of put options written on the basket of stocks that they posses. Furthermore, spread options, that are a special case of basket options, are also among the most popular products in the over the counter markets.

The pricing of such instruments has relied mostly on the use of multivariate Monte Carlo simulation methods. Binomial trees have not been deemed to be an economically appropriate method for the pricing of Basket of options since the number of nodes grows exponentially with the number of assets in the basket. On the other hand, Monte Carlo simulation presents the advantage that the number of operations involved in the calculation of the basket option price grows only linearly with the number of assets in the basket.

Analytical expressions for these contracts cannot be derived because basket options' payoff depends on the weighted sum of lognormal variables the distribution of which cannot be described in closed form. The first attempt for an analytical valuation of these instruments was made by Gentle (1993). He approximated the weighted average of the underlying assets that comprise the basket with the respective geometric average.

The geometric weighted average of lognormal variables is also a lognormal variable and a closed form expression can be easily derived for an option with such a payoff. Huynh (1994) further developed Gentle's (1993) methodology by assuming that the distribution of the weighted sum of the variables underlying the basket can be expressed as an Edgeworth series expansion around a known lognormal distribution.

As with the case of Asian options both of these approaches were under the assumption that the underlying assets all follow correlated Geometric Brownian Motions, which is equivalent to assuming that the terminal risk neutral distribution of the underlying assets is lognormal.

In this chapter a methodology similar to the one presented in Chapter 2 will be developed and basket options will be priced within the more realistic framework of a Bernoulli jump diffusion model. Specifically, within the aforementioned framework closed form expressions will be derived for basket options written on two assets. The results will be subsequently generalised for the n-asset case and the methodology will be extended with the use of Edgeworth series expansions in order to obtain even more accurate results.

7.1 The Pricing Framework (The two-asset case)

The framework for the pricing of a basket option written on two assets will be demonstrated in this section. The payoff of a basket option is expressed with the following formula:

$$Payoff = \max(w_1 S_1(T) + w_2 S_2(T) - K, 0) \qquad \textbf{(1)}$$

where in expression (1), T is the time to maturity of the option. The Stochastic Differential Equation (SDE) that describes the dynamics of the underlying asset in this case is shown below:

$$\frac{dS_i}{S_i} = (r - d_i - \lambda_i E(k_i))dt + \sigma_i dZ_{it} + k_i dq_i \tag{2}$$

,where r is the interest rate, d_i is the dividend yield, σ_{wi} is the instantaneous variance of the underlying asset returns conditional on no jumps occurring, λ_i is the Bernoulli parameter and k_i is the random percentage jump conditional on the Bernoulli event occurring, where $1+k_i$ is lognormally distributed:

$$\ln(1+k_i) \sim N(\gamma_i - \tfrac{1}{2}\delta_i^2, \delta_i^2) \equiv N(\gamma_i', \delta_i^2), E(k) \equiv \bar{k}_i = e^{\gamma_i} - 1 \tag{3}$$

It is further assumed that the jump component is independent from the Brownian Motion that describes the arrival of new information in the market. According to this assumption and equation (2) the terminal price of each underlying asset can be written as:

$$S_i(T) = \frac{S_i}{1 + \lambda_i TE[k_i]} \exp((r - d_i - \tfrac{1}{2}\sigma_i^2)T + \sigma_i Z_i(T))Y_i \tag{4}$$

In equation (4) Y_i is the product of a Bernoulli and a lognormal independent variable. Specifically Y_i takes the value of $(1+k)$ with probability λ_iT or takes the value 1 with probability $1-\lambda_i T$. From this we can deduct that the expected value of Y_i will be given by the following expression:

$$E(Y_i) = \lambda_i T(1 + E[k_i]) + 1 - \lambda_i T = 1 + \lambda_i TE[k_i] \tag{5}$$

From expressions (5) and (4) it can be understood that the adjustment in the drift present in equation (2) was introduced in order to recover the forward price of the asset as its expected value under the risk neutral measure.

This is illustrated in formula (6) below

$$E[S_i(T)] = E\left[\frac{S_i}{1+\lambda_i TE[k_i]} e^{(r-d_i-\frac{1}{2}\sigma_i^2)T+\sigma_i Z_i(T)} Y_i\right] \Rightarrow$$
$$E[S_i(T)] = \frac{S_i}{1+\lambda_i TE[k_i]} e^{(r-d_i-\frac{1}{2}\sigma_i^2)T} E\left[e^{\sigma_i Z_i(T)}\right] E[Y_i] \Rightarrow \tag{6}$$
$$E[S_i(T)] = \frac{S_i}{1+\lambda_i TE[k_i]} e^{(r-d_i)T}(1+\lambda_i TE[k_i]) = S_i e^{(r-d_i)T}$$

Since the study involves more than one assets further assumptions need to be made. The two Brownian motion components of the assets' law of motion are assumed to be correlated with correlation coefficient ρ:

$$E[dZ_1(t)dZ_2(t)] = \rho t \tag{7}$$

The two Bernoulli components of the assets' law of motion are assumed to be independent of each other and independent of the respective Brownian motions.

These additional two assumptions complete the framework within which basket options on two assets will be priced.

7.2 Pricing Methodology

The payoff of a basket option from (1) can be written as:

$$Payoff = \max(w_1 S_1(T) + w_2 S_2(T) - K, 0)$$

In order to calculate the value of the option the distribution of the weighted sum of the two assets needs to be determined and then the risk neutral valuation principle has to be used to evaluate the option price as the discounted expected risk neutral payoff.

The method that will be utilised in order to obtain the risk neutral price of the basket option resembles the method used in the previous chapter for Asian options. The final value $S_i(T)$ is given from formula (4).

$$S_i(T) = \frac{S_i}{1+\lambda_i TE[k_i]}\exp((r - d_i - \tfrac{1}{2}\sigma_i^2)T + \sigma_i Z_i(T))Y_i$$

Because of the intractable term Y_i appearing in the formula it is best to condition on the random event happening or not and work with the conditional variable. By conditioning on the Bernoulli event we obtain a variable that is a product of two other lognormal variables, namely, the Brownian motion appearing in the exponential and the jump size. Being a product of two lognormal variables this is simply another lognormal variable.

If the event of whether a jump occurs for the i-th asset is denoted in the following way:

J_i={The jump occurs for the ith asset}

Then, as previously argued, the variable S_i/J_i is a lognormal variable and it is:

$$E(S_i / J_i) = \frac{S_i}{1+\lambda_i TE[k_i]}e^{(r-d_i)T}e^{\gamma_i} \quad \textbf{(8)}$$

where γ_i is taken from equations (3). Also from (3) it can be easily seen that:

$$E[k_i] = e^{\gamma_i} - 1 \quad \textbf{(9)}$$

Applying the same rational the expected value of S_i conditioning on the jump event not happening is equal to:

$$E(S_i / J_i^{'}) = \frac{S_i}{1+\lambda_i TE[k_i]}e^{(r-d_i)T} \quad \textbf{(10)}$$

Nevertheless, there is another complexity that has to be dealt with. Even after the conditioning, and although each conditional variable is lognormal, when one wishes to evaluate the distribution of the sum of two lognormal variables there is no known closed form expression for the distribution for that sum. At this point an approximation needs to be made if a closed form solution is to be obtained for the option price. Later it will be shown how this approximation can be improved by the use of an Edgeworth series expansion while still being able to obtain closed form expression for the basket options.

It will be assumed that the distribution of the sum of lognormal variables is also a lognormal variable since there is a large body of evidence suggesting that the distribution of the sum of correlated lognormal variables is well approximated by another lognormal[1] variable. The variable is named *Bskt*, with unknown mean *a(T)* and unknown variance *v(T)* then these two parameters can be determined by considering the moment generating function for *Bskt* given by:

$$E[\,Bskt\,]^k = e^{ka(T)+0.5k^2 v(T)^2} \tag{11}$$

Analytically, *Bskt* is defined as the weighted sum of the two asset terminal values conditional on the jump occurring or not:

As an example the variable *Bskt* conditioning on the jump occurring during the lifetime of the option for the first and not occurring for the second asset is given by the following expression:

$$Bskt = S_1 / J_1 + S_2 / J_2^{'} \tag{12}$$

In order to calculate the quantities *a(T) a*nd *v(T)* for each possible definition of the variable *Bskt,* the first and second moments of the RHS of equation (12) are computed and are set equal to the respective quantities of the *Bskt* variable. Solving equations (11) for *k*=1,2, for the quantities *a(T)* and *v(T)* we find:

$$\begin{aligned} a(T) &= 2\ln E[Bskt] - \tfrac{1}{2}\ln E[Bskt^2] \\ v(T) &= \sqrt{\ln E[Bskt^2] - 2\ln E[Bskt]} \end{aligned} \tag{13}$$

[1] See Fenton (1960). Naus (1969). Janos (1970),.Hamdan (1971).Barakat (1976).

Returning to the basket option valuation problem the value of the Basket call option can be written again as:

$$Basket = e^{-rT} E[\max(w_1 S_1(T) + w_2 S_2(T) - K, 0)] \tag{14}$$

Naming B_2 the variable:

$$B_2 = w_1 S_1(T) + w_2 S_2(T) \tag{15}$$

It is:

$$B_2 = B_2 / J_1 J_2 + B_2 / J'_1 J_2 + B_2 / J'_2 J_1 + B_2 / J'_1 J'_2 \tag{16}$$

Each term in the RHS of equation (16) is a lognormal variable that corresponds to a definition of the variable *Bskt* defined previously. In order to calculate the value of the option from (14) the distribution of B_2 should be recovered. Being a sum of lognormal variables it will assumed that B_2 is also lognormal and its first and second moments will be computed from equation (16) and consequently its distribution will be fully described.

The expected value of B_2 is given by:

$$E(B_2) = E(B_2 / J_1 J_2) P(J_1 J_2) + E(B_2 / J'_1 J_2) P(J'_1 J_2) + \\ + E(B_2 / J'_2 J_1) P(J'_2 J_1) + E(B_2 / J'_1 J'_2) P(J'_1 J'_2)$$

This can be simplified to:

$$E(B_2) = w_1 E(S_1) + w_2 E(S_2) = \\ = w_1 E(S_1 / J_1) P(J_1) + w_1 E(S_1 / J'_1) P(J'_1) + w_2 E(S_2 / J_2) P(J_2) + w_2 E(S_2 / J'_2) P(J'_2) \tag{17}$$

Equation (17) can be written in a more compact form in the following manner:

$$E(B_2) = \sum_{i=1}^{2} w_i [E(S_i / J_i) P(J_i) + E(S_i / J'_i) P(J'_i)] \tag{18}$$

From equation (18) and equations (8) and (10) the expected value of B can be easily found to be:

$$E(B_2) = \sum_{i=1}^{2} w_i \left\{ \lambda_i T \frac{S_i}{1 + \lambda_i T E[k_i]} e^{(r-d_i)T} e^{\gamma_i} + (1 - \lambda_i T) \frac{S_i}{1 + \lambda_i T E[k_i]} e^{(r-d_i)T} \right\}$$

To simplify the above expression we define the quantity:

$$Sdis_i = \frac{S_i}{1+\lambda_i TE[k_i]} e^{(r-d_i)T} \tag{19}$$

Rewriting the equation above it is now:

$$E(B_2) = \sum_{i=1}^{2} w_i \left\{ \lambda_i TSdis_i e^{\gamma_i} + (1-\lambda_i T) Sdis_i \right\} \tag{20}$$

with $E[k_i]$ taken from equation (9).

The same methodology is followed in order to recover the second moment of the variable B_2. It is:

$$B_2{}^2 = \sum_{i=1}^{2} w_i{}^2 S_i^2 + 2w_1 w_2 S_1 S_2 = (I) + (II) \tag{21}$$

Term *(I)* in equation (21) represents the sum of the squared terms and term *(II)* the cross product term. Each term in *(I)* can be analysed as in equation (18) the only difference being that there will now be squared terms in the expectations. The expectation of $S_i^2(T)$ for example, given that there was a jump during the life of the option will now be given from:

$$E[S_i(T)^2] = E\left\{ \frac{S_i}{1+\lambda_i TE[k_i]} \exp((r-d_i-\tfrac{1}{2}\sigma_i^2)T + \sigma_i Z_i(T)) \right\}^2 \cdot E[1+k]^2$$

By replacing to the above expression equation (19) it becomes:

$$E[S_i(T)^2] = Sds_i{}^2 e^{\sigma_i^2 T} e^{2\gamma_i + \delta_i^2} \tag{22}$$

Combining equations (19), (20) and (22) the term *(I)* of equation (21) becomes:

$$E[(I)] = \sum_{i=1}^{2} w_i^2 Sds_i{}^2 e^{\sigma_i^2 T} (\lambda_i Te^{2\gamma_i + \delta_i^2} + 1 - \lambda_i T) \tag{23}$$

The expectation of the second term of equation (21) has a more complex analysis since in the cross product both assets appear and the correlation structure is expected to play a significant role. The expectation of term *(II)* can be analysed as:

$$\begin{aligned} E(II) = 2w_1w_2\{ & E(S_1S_2 / J_1J_2)P(J_1J_2) + E(S_1S_2 / J_1J_2^{'})P(J_1J_2^{'}) + \\ & + E(S_1S_2 / J_1^{'}J_2)P(J_1^{'}J_2) + E(S_1S_2 / J_1^{'}J_2^{'})P(J_1^{'}J_2^{'}) \end{aligned} \quad \textbf{(24)}$$

Naming each of the four terms in (24) as *(a)*, *(b)*, *(c)* and *(d)* respectively their values can be computed in the following manner.

$$(a) = E(S_1S_2 / J_1J_2)P(J_1J_2) = Sds_1Sds_2e^{-\frac{1}{2}(\sigma_1^2+\sigma_2^2)T}E[e^{\sigma_1Z_1(T)+\sigma_2Z_2(T)}]\lambda_1T\lambda_2Te^{\gamma_1+\gamma_2} \quad \textbf{(25)}$$

The expectation term in the above equation is the expected value of a lognormal variable, since both terms in the exponent are normal and the sum of normal variables is also a normal variable. If the variable of the sum is named SZ then it is:

$$SZ = \sigma_1Z_1(T) + \sigma_2Z_2(T) \Rightarrow$$
$$E[SZ] = 0$$

and

$$Var[SZ] = E[SZ^2] = \sigma_1^{\,2}E[Z_1(T)^2] + \sigma_2^{\,2}E[Z_2(T)^2] + 2\sigma_1\sigma_2\operatorname{cov}(Z_1(T), Z_2(T)) \Rightarrow$$
$$Var[SZ] = \sigma_1^{\,2}T + \sigma_2^{\,2}T + 2\sigma_1\sigma_2\rho\sqrt{T}\sqrt{T} = (\sigma_1^{\,2} + \sigma_2^{\,2} + 2\sigma_1\sigma_2\rho)T$$

Therefore:

$$E[e^{\sigma_1Z_1(T)+\sigma_2Z_2(T)}] = E[e^{SZ}] = e^{E[SZ]+\frac{1}{2}Var[SZ]^2} = e^{\frac{1}{2}(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)T} \quad \textbf{(26)}$$

From equations (25) and (26) the value of *(a)* is:

$$\begin{aligned} (a) &= Sds_1Sds_2e^{-\frac{1}{2}(\sigma_1^2+\sigma_2^2)T}\lambda_1T\lambda_2Te^{\gamma_1+\gamma_2}e^{\frac{1}{2}(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)T} \Rightarrow \\ (a) &= Sds_1Sds_2\lambda_1T\lambda_2Te^{\gamma_1+\gamma_2}e^{\rho\sigma_1\sigma_2T} \end{aligned} \quad \textbf{(27)}$$

Applying the exact same arguments the values of (b), (c) and (d) can be calculated as follows:

$$(b) = Sds_1Sds_2\lambda_1T(1-\lambda_2T)e^{\gamma_1}e^{\rho\sigma_1\sigma_2T} \quad \textbf{(28)}$$

$$(c) = Sds_1Sds_2(1-\lambda_1T)\lambda_2Te^{\gamma_2}e^{\rho\sigma_1\sigma_2T} \quad \textbf{(29)}$$

$$(d) = Sds_1 Sds_2 (1-\lambda_1 T)(1-\lambda_2 T)e^{\rho\sigma_1\sigma_2 T} \tag{30}$$

Combining equations (27), (28), (29) and (30) we are able to compute analytically the value of the expectation of the second term in equation (21) from (24). Finally, the quantity $E[B_2{}^2]$ can easily be computed as a combination of (24) and (23).

The variable B_2 will be assumed to be a lognormal variable as an approximation, as it was previously mentioned. As in equation (11) the mean and variance *a(T)* and *v(T)* respectively of the corresponding normal variable can be computed through equations (13).

Having computed the first and second moments of the variable B_2 the calculation of the value of the basket option is equivalent to the valuation of a call option on an asset the terminal risk neutral density of which is a lognormal distribution with parameters *a(T)* and *v(T)*. The value of this option is given by the formula:

$$Basket = e^{-rT} E[\max(B_2 - K,0)] = e^{-rT}\{e^{a(T)+\frac{1}{2}v^2(T)}N(d_1) - XN(d_2)\} \tag{31}$$

with $d_1 = \dfrac{-\ln(K) + a(t) + v(t)^2}{v(t)}$,

$d_2 = d_1 - v(t)$

Formula (31) gives the final value a basket call option within a Bernoulli jump diffusion framework. For put option valuation expression (31) changes since the expected payoff changes. It is:

$$Basket = e^{-rT} E[\max(K - B,0)] = e^{-rT}\{XN(-d_2) - e^{a(T)+\frac{1}{2}v^2(T)}N(-d_1)\} \tag{32}$$

,where the d_1 and d_2 are as before.

This completes the valuation of a two-asset basket option when the two underlying assets follow a Bernoulli jump diffusion process. Next the generalisation follows for the n-asset case.

7.3 The Pricing Framework (The n-asset case)

The framework for the pricing of a basket option written on n assets will be demonstrated in this section. The payoff of a basket option is expressed by the following formula:

$$Payoff = \max(\sum_{i=1}^{n} w_i S_i(T) - K, 0) \tag{33}$$

where in expression (33), T is the time to maturity of the option. The Stochastic Differential Equation (SDE) that describes the dynamics of each underlying asset in this case is shown below:

$$\frac{dS_i}{S_i} = (r - d_i)dt - \ln(1 + \lambda_i dtE(k_i)) + \sigma_i dZ_{it} + k_i dq_i$$

,where r is the interest rate, d_i is the dividend yield, σ_{wi} is the instantaneous variance of the underlying asset returns conditional on no jumps occurring, λ_i is the Bernoulli parameter and k_i is the random percentage jump conditional on the Bernoulli event occurring, where $1+k_i$ is lognormally distributed:

$$\ln(1 + k_i) \sim N(\gamma_i - \tfrac{1}{2}\delta_i^2, \delta_i^2) \equiv N(\gamma_i', \delta_i^2), E(k) \equiv \bar{k}_i = e^{\gamma_i} - 1, \quad i = 1,...,n \tag{34}$$

It is further assumed that the jump component is independent from the Brownian Motion that describes the arrival of new information in the market. The Brownian motion components of the assets' law of motion are assumed to be correlated and the individual components of the correlation coefficient matrix are given by the following relations:

$$E[dZ_i(t)dZ_j(t)] = \rho_{ij}t \quad \textbf{(35)}$$

The two Bernoulli components of the assets' law of motion are assumed independent of each other and independent of the respective Brownian motions as in the two-asset case.

7.3.1 Pricing Methodology

The pricing methodology is a generalisation of the one suggested for the two-asset case. The objective is to recover the distribution of the variable that represents the weighted sum of the *n* assets.

Let this variable be:

$$Bn = \sum_{i=1}^{n} w_i S_i(T)$$

In order to determine the risk neutral distribution of the variable *Bn* it will be necessary to condition on each of the *n* random jumps occurring or not during the lifetime of the option. The conditional variables are lognormal and they are given by equation (8) as before.

The summation variable *Bn* has a more complex analysis as a summation of lognormal variables than the analysis in equation (16). Despite the complexity, though, it still is simply a summation of lognormal variables. Thus, it will assumed that *Bn* is a lognormal variable as well, as an approximation and its first and second moments will be calculated analytically. Consequently the parameters of the corresponding normal distribution will be recovered through equations (13) and the basket call and put options will be priced through equations (31) and (32) respectively.

The first moment of *Bn* can be calculated through the following equation:

$$E[Bn] = E\left[\sum_{i=1}^{n} w_i S_i(T)\right] = \sum_{i=1}^{n} w_i E[S_i(T)] =$$
$$= \sum_{i=1}^{n} w_i \{E[S_i(T)/J_i]P(J_i) + E[S_i(T)/J_i^{'}]P(J_i^{'})\}$$

where the quantities $E[S_i(T)/J_i]$ and $E[S_i(T)/J_i^{'}]$ can be calculated from equations (8) and (10) respectively. Therefore, it is:

$$E[Bn] = \sum_{i=1}^{n} w_i Sds_i \{\lambda_i T e^{\gamma_i} + 1 - \lambda_i T\} \quad \textbf{(36)}$$

Expression (36) represents the value of the first moment of the summation variable *Bn*. The expression for the second moment of the variable is slightly more complicated.

$$E[Bn^2] = E[\sum_{i=1}^{n} w_i S_i(T)]^2 = \sum_{i=1}^{n} w^2{}_i E[S_i(T)]^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} w_i w_j E[S_i(T) S_j(T)] \quad \textbf{(37)}$$

Equation (37) is exactly similar to equation (21). The first summation term in the RHS of (37) contains the same terms with the term *(I)* of (21). Thus, naming the first term in (37) (I_{Bn}) the following equation can be easily derived:

$$E[(I_{Bn})] = \sum_{i=1}^{n} w_i^2 Sds_i{}^2 e^{\sigma_i^2 T} (\lambda_i T e^{2\gamma_i + \delta_i^2} + 1 - \lambda_i T) \quad \textbf{(38)}$$

Each expectation in the second double summation term in the RHS of (37) can be in turn analysed in four different terms in a similar manner as in equation (24). Therefore by naming the general term as (II_{Bn}) it is:

$$E(II_{Bn}) = 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} w_i w_j \{E(S_i S_j / J_i J_j) P(J_i J_j) + E(S_i S_j / J_i J_j^{'}) P(J_i J_j^{'}) + \quad \textbf{(39)}$$
$$+ E(S_i S_j / J_i^{'} J_j) P(J_i^{'} J_j) + E(S_i S_j / J_i^{'} J_j^{'}) P(J_i^{'} J_j^{'})\}$$

The value of each term in the RHS of equation (39) is given in equations (27) to (30) with the order that they appear in the summation. Having estimated all the quantities needed for the calculation of $E[Bn]^2$ a comprehensive formula can be given that combines all of them in one equation.

$$E[Bn]^2 = (38) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} w_i w_j \{(27)+(28)+(29)+(30)\} \quad \textbf{(40)}$$

Finally, to calculate the option value of the basket on n assets the exact same arguments as in the two-asset case are applied. Specifically, as it was mentioned in the beginning of the section, since Bn is assumed to be a lognormal variable, its parameters are recovered through equations (13), (36) and (40).

$$a(T) = 2\ln E[Bn] - \tfrac{1}{2}\ln E[Bn^2]$$
$$v(T) = \sqrt{\ln E[Bn^2] - 2\ln E[Bn]}$$

The value of a call option written on the Basket with multiple assets can be calculated from equation (31), using as inputs the parameters of the above equation, whereas this of a put option can be calculated from equation (32).

7.4 Simulation Results

The previous two sections described a new methodology for the pricing of Basket options. A Bernoulli jump diffusion process is suggested as a more realistic process than the Geometric Brownian for the description of the movements of the underlying asset. The former process has been proven to describe more accurately the law of motion of a variety of underlying asset as it was argued extensively in the previous chapter. Nevertheless, since all transactions including Basket options take place in the Over the Counter (OTC) market there is no way to directly test the method suggested in this chapter against market data. A formal test of the method then, would be against the assumptions of the method itself.

The method provides closed form solutions for the prices of European call and put Basket options with the assumption, though, that the sum of lognormal variables is also a lognormal variable. This assumption is the only source of possible errors in the model since all the rest follow through mathematical derivations. One should wish to test this assumption in order to validate the results provided by the method.

The best way to achieve this goal is through Monte Carlo simulation. The benchmark prices will be produced with the assumption that all assets that underlie the Basket follow a Bernoulli jump diffusion stochastic process as described in the pricing framework section of the analysis. This approach will enable a control of whether the prices produced through the analytic expressions are in accordance with the theoretical assumptions of the model. This will test in effect the validity of the approximation of a sum of lognormal variables with another lognormal variable.

The test was performed only for the two-asset case since the more variables that are included in the sum the more the sum converges to a lognormal variable[2]. Therefore the greatest discrepancies between theoretical and analytical prices were expected to be found in the two-asset case.

The Monte Carlo experiment was set up in the following way. The option formula tested was that of a call option on two assets. The initial value of both assets was set at 100, the time to maturity of the call basket option was assumed to be one year and a common interest rate was assumed of 7%. Furthermore, the first asset was assumed to have a dividend yield of 1% while the second asset a dividend yield of 2%. Various combinations of the strike price, the individual volatilities and the expected jump sizes were used as inputs to the MC simulation. The standard deviation of the jump sizes though was kept fixed at values equal to 0.005 and 0.007 for the first and second asset respectively. Also, the parameters λ_1, λ_2 were set at 0.2 and 0.3 corresponding to jump probabilities of $P(J_1)= \lambda_1 T=0.2$ and $P(J_2)= \lambda_2 T=0.3$ respectively. Finally equal weights were assumed for the two assets comprising the basket and the correlation coefficient was set at the value of 0.4[3].

In the tables that follow the recovered values through Monte Carlo simulation and the ones postulated by the model are presented. The variables my_1 and my_2 correspond to the expected jump sizes of the jump components for the first and second asset respectively. The standard error is for 100,000 simulations and it is reported as a percentage of the option price. Standard errors were approximately the same for every choice of asset volatilities and therefore, only one value is being reported per table. The strike is reported as a ratio of the weighted spot price.

[2] See Fenton (1960), Janos (1970),.Barakat (1976)
[3] All the parameter values were randomly chosen. Different combinations were also tried producing exactly similar results.

Table 1: Simulation Results for vol_1=0.3, vol_2=0.2

		7.4.1.1	M	MC		MC	7.4.1.1.2.1	MC	7.4.1.1.2.1.1
		my_1=0.01, my_2=0.02		my_1=0, my_2=-0.02		my_1=0.02, my_2=0.02		my_1=0.02, my_2=-0.02	
		6.2331	6.23372	6.23267	6.23327	6.23449	6.2349	6.23551	6.23485
vol_1=0.3	*/K=0.95*	8.49004	8.49077	8.48947	8.49032	8.4922	8.49196	8.49135	8.4919
vol_2=0.2	*/K=1*	11.0079	11.0074	11.0064	11.0069	11.0096	11.0085	11.0082	11.0084
	/K=1.05	13.6942	13.694	13.6936	13.6936	13.6943	13.695	13.6948	13.695
	/K=1.1	16.4714	16.472	16.4712	16.4716	16.473	16.4728	16.4735	16.4728
Standard Error: 4.16E-05									

Table 2: Simulation Results for vol_1=0.3, vol_2=0.1

		7.4.1.2	M	MC		MC	7.4.1.2.2.1	MC	7.4.1.2.2.1.1
		my_1=0.01, my_2=0.02		my_1=0, my_2=-0.02		my_1=0.02, my_2=0.02		my_1=0.02, my_2=-0.02	
		4.9242	4.92223	4.92136	4.92169	4.92357	4.92364	4.92373	4.92357
vol_1=0.3	*/K=0.95*	7.1707	7.17104	7.17079	7.17049	7.17201	7.17247	7.17207	7.1724
vol_2=0.1	*/K=1*	9.77122	9.77121	9.77124	9.7707	9.77266	9.77254	9.77258	9.77248
	/K=1.05	12.6046	12.6049	12.604	12.6045	12.606	12.6061	12.6055	12.606
	/K=1.1	15.5613	15.5614	15.5608	15.5611	15.5619	15.5624	15.562	15.5623
Standard Error: 6.67E-0.5									

Table 3: Simulation Results for vol_1=0.2, vol_2=0.2

		7.4.1.3	M	MC		MC	7.4.1.3.2.1	MC	7.4.1.3.2.1.1
		my_1=0.01, my_2=0.02		my_1=0, my_2=-0.02		my_1=0.02, my_2=0.02		my_1=0.02, my_2=-0.02	
		4.50846	4.50854	4.50754	4.50799	4.51128	4.50997	4.50991	4.5099
vol_1=0.2	*/K=0.95*	6.75067	6.75105	6.74985	6.75049	6.7523	6.7525	6.75212	6.75242
vol_2=0.2	*/K=1*	9.38243	9.38196	9.38175	9.38145	9.38333	9.38331	9.38394	9.38324
	/K=1.05	12.271	12.2712	12.2704	12.2708	12.2719	12.2723	12.2723	12.2723
	/K=1.1	15.2935	15.2938	15.2938	15.2935	15.295	15.2947	15.2954	15.2947
Standard Error:4.67E-0.5									

From Tables 1, 2 and 3, it becomes evident that the closed form expressions that were derived under the assumption that the sum of two lognormal variables is also a lognormal variable can provide accurate pricing formulae for Basket options on two assets. The highest percentage differences between the Monte Carlo estimates and the analytical expressions across all different strikes, volatilities and expected jump sizes were less than 0.04%.

Obtaining a closed form solution for the prices of Basket options on several assets does provide practitioners with a significant advantage since Monte Carlo simulation can be avoided. MC can be significantly time consuming and computationally intensive if an acceptable degree of accuracy is to be achieved, since the standard error of the estimate is reduced only with the squared root of the number of simulations. The advantage of this method is even greater when Basket options with more than two underlying assets are examined since the computational cost grows linearly with the number of variables in a Monte Carlo experiment.

7.5 BASKET GREEKS (The n-asset case)

One of the most significant reasons why an analytic solution is preferable to a simulation solution to a valuation problem is that one can observe the functional form that contains all the observed and unobserved variables and their way of interacting can be observed and comprehended. Furthermore, the sensitivity of the derivative contract with respect to these variables can also be estimated. The sensitivities are the hedging parameters or "Greeks" of the derivative. Their estimation from a closed form expression is a straightforward procedure while if one had to rely on Monte Carlo methods in order to recover these parameters, a less accurate estimate would have been generated.

For ease of reference equations (31) and (32) are rewritten below.

$$Basket = e^{-rT}E[\max(Bn-K,0)] = e^{-rT}\{e^{a(T)+\frac{1}{2}v^2(T)}N(d_1) - XN(d_2)\} \quad \textbf{(41)}$$

which is the value of a Basket call option and

$$Basket = e^{-rT}E[\max(K-Bn,0)] = e^{-rT}\{XN(-d_2) - e^{a(T)+\frac{1}{2}v^2(T)}N(-d_1)\} \quad \textbf{(42)}$$

which gives the value of a Basket put option.

with $d_1 = \dfrac{-\ln(K) + a(t) + v(t)^2}{v(t)}$,

$d_2 = d_1 - v(t)$

The analysis will only be carried out for call options. Put options' Deltas and Gammas can be computed in an exact similar fashion.

The expression for the first and second moments of the weighted sum of n assets are rewritten below for ease of reference.

From equation (36) the first moment of Bn is:

$$E[Bn] = \sum_{i=1}^{n} w_i Sds_i \{\lambda_i Te^{\gamma_i} + 1 - \lambda_i T\}$$

which can be simplified to:

$$E[Bn] = \sum_{i=1}^{n} S_i y_i \quad \textbf{(43)}$$

with y_i being:

$$y_i = \frac{w_i}{1+\lambda_i TE[k_i]} e^{(r-d_i)T}\{\lambda_i Te^{\gamma_i} + 1 - \lambda_i T\} \quad \textbf{(44)}$$

Equation (40) gives the expression for the second moment of Bn. The simplified version of it will only be presented here:

$$E[Bn^2] = \sum_{i=1}^{n} z_i S_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} y_{ij} S_i S_j \quad \textbf{(45)}$$

With

$$z_i = \left(\frac{w_i}{1+\lambda_i TE[k_i]}\right)^2 e^{2(r-d_i)T+\sigma_i^2 T}\{\lambda_i T e^{2\gamma_i+\delta_i^2}+1-\lambda_i T\}$$

and

$$y_{ij} = \left(\frac{w_i w_j}{1+\lambda_i TE[k_i]}\right)\left(\frac{e^{(r-d_i)T+(r-d_j)T}}{1+\lambda_j TE[k_j]}\right) e^{\rho\sigma_i\sigma_j T}\{\lambda_i T\lambda_j T e^{\gamma_i} e^{\gamma_j}+\lambda_i T e^{\gamma_i}(1-\lambda_j T)+$$
$$\lambda_j T e^{\gamma_j}(1-\lambda_i T)+(1-\lambda_i T)(1-\lambda_j T)\}$$

The first derivatives of the moments of *Bn* will need to be estimated for the Delta calculation and the second derivatives for the Gamma calculation. Thus expression (41) will be differentiated twice and all the partial derivatives with respect to each asset S_i will be computed.

The first derivative of the first moment of *Bn* is:

$$E'[Bn] = \frac{\partial E[Bn]}{\partial S_i} = y_i \tag{46}$$

and the second can easily found to be:

$$E''[Bn] = 0 \tag{47}$$

The first derivative of the second moment of *Bn* is:

$$E'[Bn^2] = \frac{\partial E[Bn^2]}{\partial S_i} = 2S_i z_i + 2\sum_{j=1, j\neq i}^{n} S_j y_{ij} \tag{48}$$

The second derivative of the second moment of *Bn* is given by the expression:

$$E''[Bn^2] = \frac{\partial^2 E[Bn^2]}{\partial S_i^{\,2}} = 2z_i \tag{49}$$

Furthermore the derivatives of the quantities *a(T)* and *v(T)* have to be computed: It is:

$$a' = a'(T) = \frac{\partial a(T)}{\partial S_i} = 2\frac{E'[Bn]}{E[Bn]} - \frac{1}{2}\cdot\frac{E'[Bn^2]}{E[Bn^2]} \tag{50}$$

and

$$a'' = a''(T) = \frac{\partial^2 a(T)}{\partial S_i^{\,2}} = -2\frac{(E'[Bn])^2}{E^2[Bn]} - \frac{1}{2}\cdot\frac{(E'[Bn^2])^2 - E[Bn^2]\cdot E''[Bn^2]}{E^2[Bn^2]} \quad \textbf{(51)}$$

For $v(T)$ the first and second derivatives are given from the equations below:

$$v' = v'(T) = \frac{\partial v(T)}{\partial S_i} = \frac{\dfrac{E'[Bn^2]}{E[Bn^2]} - \dfrac{E'[Bn]}{E[Bn]}}{2\sqrt{\ln E[Bn^2] - 2E[Bn]}} \quad \textbf{(52)}$$

and

$$v'' = v''(T) = \frac{\partial^2 v(T)}{\partial S_i^{\,2}} = \frac{-1}{4}\frac{v'^2}{v^3} + \frac{\dfrac{(E'[Bn^2])^2 - E[Bn^2]\cdot E''[Bn^2]}{E^2[Bn^2]} - 2\dfrac{(E'[Bn])^2}{E^2[Bn]}}{v} \quad \textbf{(53)}$$

Finally the derivatives of d_1 and d_2 with respect to $a(T)$ and $v(T)$ have to be computed.

For d_1 it is:

$$d_1{}' = \frac{\partial d_1}{\partial S_i} = \frac{a'v - v'(a - \ln K)}{v^2} + v' \quad \textbf{(54)}$$

and

$$d_1{}'' = \frac{\partial^2 d_1}{\partial S_i^{\,2}} = \frac{v[a''v - v''(a - \ln K)] - 3v[a'v - v'(a - \ln K)]}{v^3} + v''$$

Similarly for d_2 it is:

$$d_2{}' = \frac{\partial d_2}{\partial S_i} = d_1{}' - v'$$

and **(55)**

$$d_2{}'' = \frac{\partial^2 d_2}{\partial S_i^{\,2}} = d_1{}'' - v''$$

The partial delta of each assets included in the basket is defined as the partial derivative of the option price with respect to the particular asset's price. Therefore, for each Delta it is:

$$Delta_i = \frac{\partial C}{\partial S_i} = \frac{\partial[e^{-rT}\{e^{a(T)+\frac{1}{2}v^2(T)}N(d_1) - XN(d_2)\}]}{\partial S_i} = e^{-rT}\frac{\partial[E[Bn]N(d_1) - XN(d_2)]}{\partial S_i}$$

$$\Rightarrow Delta_i = e^{-rT}\{E'[Bn]N(d_1) + E[Bn]N'(d_1)d_1' - XN'(d_2)d_2'\} \qquad \textbf{(56)}$$

where in the expression (56):

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Equation (56) completes the calculation of the partial delta for a basket option written on n assets. Differentiating expression (56) once more an expression for the partial Gamma of each asset included in the basket, is obtained. Analytically:

$$\begin{aligned} Gamma_i &= \frac{\partial^2 C}{\partial S_i^2} = \frac{\partial Delta_i}{\partial S_i} = \\ &= e^{-rT}\{E''[Bn]N(d_1) + E[Bn]'N'(d_1)d_1' + E[Bn][N''(d_1)(d_1')^2 + N'(d_1)d_1''] \\ &\qquad - X[N''(d_2)(d_2')^2 + N'(d_2)d_2'']\} \end{aligned} \qquad (57)$$

where in expression (57)

$$N''(x) = \frac{-x}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

Equation (57) contains the partial Gamma for each of the n assets contained in the underlying basket. The Greek parameters estimated in this section will enable practitioners to actively hedge the exotic products within the Bernoulli jump diffusion framework.

7.6 Edgeworth Series Expansion

The results reported in Tables 1, 2 and 3 prove that the suggested method can provide adequately accurate results for the values of basket options on multiple assets. However, for reasons of completeness the Edgeworth series expansion method will also be examined in this section. Thus, in this section the distribution of the sum of n lognormal variables will be approximated around a lognormal distribution and closed form expressions for basket options will again be derived.

Huynh (1994) applied a similar methodology to Basket options but within a Geometric Brownian Motion framework generalising Gentle' (1994) methodology and Turnbull and Wakeman (1991) also used an Edgeworth series expansion to express the distribution of the sum of correlated lognormal variables in a study on Asian options generalising the approach of Levy (1992). In the previous chapter, it was demonstrated how Edgeworth series expansions can be used within a Bernoulli Jump diffusion framework for the case of Asian options.

An Edgeworth series expansion is the probabilistic analogue of a Taylor series expansion. The distribution function of a variable is approximated with the help of its cumulants. As in Taylor series expansion the function is approximated around a specific point x_0, here the distribution function of a random variable is approximated around an *a priori* given approximating distribution. So if the true probability function is denoted by $f(y)$ and the approximating one by $a(y)$ then:

$$f(y) = a(y) + \frac{c_2}{2!}\frac{d^2 a(y)}{dy^2} - \frac{c_3}{3!}\frac{d^3 a(y)}{dy^3} + \frac{c_4}{4!}\frac{d^4 a(y)}{dy^4} + e(y) \qquad \textbf{(58)}$$

,where $c_2 = \chi_2(F) - \chi_2(A); c_3 = \chi_3(F) - \chi_3(A); c_4 = \chi_4(F) - \chi_4(A) + 3c_2^2;$

and $\chi_j(F)$ $[\chi_j(A)]$ is the j^{th} cumulant of the exact [approximating] distribution, and $e(y)$ is residual error term. If a random variable Y has a cumulative distribution F, the first four cumulants are:

$\chi_1(F)=E(Y)$ $\qquad$ $\chi_2(F)=E(Y-E(Y))^3$

$\chi_3(F)=E(Y-E(Y))^3$ $\chi_4(F)=E(Y-E(Y))^4-3E(Y-E(Y))^3$

where all expectations are with respect to the distribution F. From the above equations it is easily understood that in order to be able to evaluate a European Basket option the quantities $E(Y^m)$ have to be calculated for m=1,2,3,4.

Variable Y in the case of Baskets is the variable Bn. It was previously argued that Bn can be decomposed into a sum of lognormal variables. Therefore, its distribution rather than being a simple lognormal, will be expressed as an Edgeworth series expansion around a lognormal approximating distribution which will be chosen appropriately.

The application of the methodology will be demonstrated for the two-asset case. The generalisation for the n-asset case is straightforward and it will only add to the complexity of the notation while no additional mathematical or financial intuition will be gained.

In the previous sections the first and second moments of the variable B_2 were computed. The expansion method requires that its third and fourth moments be additionally computed since the first and second moments of the true and approximating distribution will be set equal.

The third moment of B_2 can be written in the following way:

$$E[B_2{}^3] = \sum_{i=1}^{2} w_i E[S_i(T)^3] + 3w_1{}^2 w_2 E[S_1(T)^2 S_2(T)] + 3w_1 w_2{}^2 E[S_1(T) S_2(T)^2] \quad \textbf{(59)}$$

There are three terms [(I), (II), (III) in the order that they appear] in equation (59) that will be analysed separately.

If Z is a normal variable with mean μ and variance σ^2 then for all equations the following identity will be extensively used:

$$E[e^{kZ}] = e^{k\mu+\frac{1}{2}k^2\sigma^2} \quad \textbf{(60)}$$

Term (*I*):

$$(I)=\sum_{i=1}^{2}w_i^3E[S_i^3(T)]=\sum_{i=1}^{2}w_i^3Sdis_i^3e^{3\sigma_i^2}[\lambda_iTe^{3\gamma_i+3\delta_i^2}+1-\lambda_iT] \quad \textbf{(61)}$$

where $Sdis_i$ is from equation (19) and it is rewritten below for ease of reference:

$$Sdis_i=\frac{S_i}{1+\lambda_iTE[k_i]}e^{(r-d_i)T} \quad (19)$$

Term (*II*):

$$(II)=3w_1^2w_2E[S_1(T)^2S_2(T)]=$$
$$=3w_1^2w_2\{E(S_1^2S_2/J_1J_2)P(J_1J_2)+E(S_1^2S_2/J_1J_2')P(J_1J_2')+$$
$$+E(S_1^2S_2/J_1'J_2)P(J_1'J_2)+E(S_1^2S_2/J_1'J_2')P(J_1'J_2')\}\Rightarrow$$

$$\Rightarrow(II)=3w_1^2w_2Sdis_1^2Sdis_2e^{2\rho\sigma_1\sigma_2T+\sigma_1^2}\{\lambda_1T\lambda_2Te^{2\gamma_1+\delta_1^2+\gamma_2}+\lambda_1T(1-\lambda_2T)e^{2\gamma_1+\delta_1^2}$$
$$+(1-\lambda_1T)\lambda_2Te^{\gamma_2}+(1-\lambda_1T)(1-\lambda_2T)\} \quad \textbf{(62)}$$

Term (*III*) is exactly symmetrical to term (*II*). In order to compute (*II*) form (*III*) the only modification needed is to switch pointers in equation (62). Thus it is:

$$(III)=3w_2^2w_1Sdis_2^2Sdis_1e^{2\rho\sigma_1\sigma_2T+\sigma_2^2}\{\lambda_1T\lambda_2Te^{2\gamma_2+\delta_2^2+\gamma_1}+\lambda_1T(1-\lambda_2T)e^{\gamma_1}$$
$$+(1-\lambda_1T)\lambda_2Te^{2\gamma_2+\delta_2^2}+(1-\lambda_1T)(1-\lambda_2T)\} \quad \textbf{(63)}$$

Equations (61), (62) and (63) combined with (59) give the value for the third moment of the variable B_2.

The fourth moment of B_2 is a the fourth power of the sum in (59) and it is analysed as follows:

$$E[B_2^4]=\sum_{i=1}^{2}w_iE[S_i(T)^4]+4w_1^3w_2E[S_1(T)^3S_2(T)]+$$
$$+6w_1^2w_2^2E[S_1(T)^2S_2(T)^2]+4w_1w_2^3E[S_1(T)S_2(T)^3] \quad \textbf{(64)}$$

Again the four terms that appear in equation (64) will be named as term (*I*), (*II*), (*III*) and (*IV*) and will be individually analysed.

Term (*I*):

$$(I)=\sum_{i=1}^{2} w_i^4 E[S_i^4(T)] = \sum_{i=1}^{2} w_i^4 Sdis_i^4 e^{6\sigma^2}[\lambda_i T e^{4\gamma_i+6\delta_i^2}+1-\lambda_i T] \tag{65}$$

Term *(II)*:

$$(II)=4w_1^3 w_2 E[S_i(T)^3 S_i(T)]$$

The expectation above can be decomposed in four terms by conditioning on the jump event occurring or not for each of the two assets as in equation (62). Therefore, (II) is equal to:

$$\begin{aligned}(II) = 4w_1^3 w_2 Sdis_1^3 Sdis_2 e^{3\rho\sigma_1\sigma_2 T}\{&\lambda_1 T\lambda_2 T e^{3\gamma_1+3\delta_1^2+\gamma_2}+\lambda_1 T(1-\lambda_2 T)e^{3\gamma_1+3\delta_1^2} \\ &+(1-\lambda_1 T)\lambda_2 T e^{\gamma_2}+(1-\lambda_1 T)(1-\lambda_2 T)\}\end{aligned} \tag{66}$$

Term (III):

$$\begin{aligned}(III) &= 6w_1^2 w_2^2 E[S_i(T)^2 S_i(T)^2] = \\ &= 6w_1^2 w_2^2 Sdis_1^2 Sdis_2^2 e^{4\rho\sigma_1\sigma_2 T}\{\lambda_1 T\lambda_2 T e^{2\gamma_1+\delta_1^2+2\gamma_2+\delta_2^2}+\lambda_1 T(1-\lambda_2 T)e^{2\gamma_1+\delta_1^2} \\ &\quad +(1-\lambda_1 T)\lambda_2 T e^{2\gamma_2+\delta_2^2}+(1-\lambda_1 T)(1-\lambda_2 T)\}\end{aligned} \tag{67}$$

Term (IV):

Term (IV) is exactly symmetrical to term (II). It can be easily recovered from (II) by simply switching the pointers 1 and 2 in all variables.

$$(IV) = 4{w_2}^3 w_1 Sdis_2^3 Sdis_1 e^{3\rho\sigma_1\sigma_2 T + 3\sigma_2^2} \{ \lambda_1 T \lambda_2 T e^{3\gamma_2 + 3\delta_2^2 + \gamma_1} + \lambda_1 T (1 - \lambda_2 T) e^{\gamma_1} + (1 - \lambda_1 T) \lambda_2 T e^{3\gamma_2 + 3\delta_2^2} + (1 - \lambda_1 T)(1 - \lambda_2 T) \} \quad \mathbf{(68)}$$

Having estimated all the quantities in (64) it is now possible to calculate the fourth moment of B_2.

The third and fourth moments of B_2 were estimated so that the true distribution of B_2 could be expanded around an approximating distribution. The latter can be any distribution that is close to the true one. The choice that will be made for the specification of the approximating distribution will be the lognormal distribution that has the same first and second moments as the true distribution as it was previously mentioned. This follows the line of research in the area of implied distributions, where the Edgeworth series expansion methods have been mostly popular, Jarrow and Rudd (1982), Corrado and Su (1997).

This implies that the approximating distribution will be the distribution over which Basket options were priced in the previous sections. This is the reason why the Edgeworth series expansion is a generalisation of the approach previously followed. The latter corresponds to an Edgeworth expansion with only two terms not being able to capture the possible differences in skewness and kurtosis that the sum of lognormal variables might have from a simple lognormal variable.

Equation (58), representing the expansion of the true around the approximating distribution, involves the third and fourth moments of the latter. The parameters of the distribution can be computed from the first and second moments of the true distribution through equation (13) that is sited again below:

$$a(T) = 2\ln E[B_2] - \tfrac{1}{2}\ln E[{B_2}^2]$$
$$v(T) = \sqrt{\ln E[{B_2}^2] - 2\ln E[B_2]} \qquad (13)$$

Its first, second, third and fourth moments are given through the formulae:

$$E[A] = e^{a(T)+\frac{1}{2}v^2(T)} \qquad E[A^2] = e^{2a(T)+2v^2(T)}$$
$$E[A^3] = e^{3a(T)+\frac{9}{2}v^2(T)} \qquad E[A^4] = e^{4a(T)+8v^2(T)} \qquad \textbf{(69)}$$

The Edgeworth series option prices are finally derived from the following formula:

$$C(Y;F) = e^{-rT}\left[C(Y,A) - \frac{(\kappa_3(F)-\kappa_3(A))}{3!}\frac{dC(K)}{dY} + \frac{(\kappa_4(F)-\kappa_4(A))}{4!}\frac{d^2C(K)}{dY^2}\right]$$

where in the above expression *C(Y,A)* is the option price under the approximating distribution, and in this case it is simply equation (31) and the first and second derivatives of (31) present are evaluated at the strike price.

The results of the Edgeworth series expansion compared to the same Monte Carlo values as the ones used in Tables 1,2 and 3 are presented below.

Table 1: Simulation Results for vol_1=0.3, vol_2=0.2 using the Edgeworth Series Expansion

		MC	Analytical	MC	Analytical	MC	Analytical	MC	Analytical
		my_1=0.01, my_2=0.02		**my_1=0, my_2=-0.02**		**My_1=0.02, my_2=0.02**		**my_1=0.02, my_2=-0.02**	
		6.2331	6.23522	6.23267	6.23137	6.23449	6.2369	6.23551	6.23405
vol_1=0.3	*/K=0.95*	8.49004	8.48877	8.48947	8.49122	8.4922	8.48946	8.49135	8.492
vol_2=0.2	*/K=1*	11.0079	11.0018	11.0064	11.0107	11.0096	11.0015	11.0082	11.0093
	/K=1.05	13.6942	13.6856	13.6936	13.6999	13.6943	13.6843	13.6948	13.6967
	/K=1.1	16.4714	16.4619	16.4712	16.4796	16.473	16.46	16.4735	16.4751
Standard Error: 4.16E-05									

Table 2: Simulation Results for vol_1=0.3, vol_2=0.1 using the Edgeworth Series Expansion

		MC	Analytical	MC	Analytical	MC	Analytical	MC	Analytical
		my_1=0.01, my_2=0.02		**my_1=0, my_2=-0.02**		**My_1=0.02, my_2=0.02**		**my_1=0.02, my_2=-0.02**	
		4.9242	4.92333	4.92136	4.92019	4.92357	4.92514	4.92373	4.92297
vol_1=0.3	*/K=0.95*	7.1707	7.16824	7.17079	7.17089	7.17201	7.16917	7.17207	7.1718
vol_2=0.1	*/K=1*	9.77122	9.76481	9.77124	9.773	9.77266	9.76484	9.77258	9.77208
	/K=1.05	12.6046	12.5963	12.604	12.6084	12.606	12.5956	12.6055	12.6059
	/K=1.1	15.5613	15.5524	15.5608	15.5659	15.5619	15.5514	15.562	15.5625
Standard Error: 6.67E-0.5									

Table 3: Simulation Results for vol_1=0.2, vol_2=0.2 using the Edgeworth Series Expansion

		MC	Analytical	MC	Analytical	MC	Analytical	MC	Analytical
		my_1=0.01, my_2=0.02		**my_1=0, my_2=-0.02**		**my_1=0.02, my_2=0.02**		**my_1=0.02, my_2=-0.02**	
		4.50846	4.50974	4.50754	4.50669	4.51128	4.51147	4.50991	4.5093
vol_1=0.2	*/K=0.95*	6.75067	6.74945	6.74985	6.75219	6.7523	6.7504	6.75212	6.75312
vol_2=0.2	*/K=1*	9.38243	9.37776	9.38175	9.38595	9.38333	9.37791	9.38394	9.38514
	/K=1.05	12.271	12.2655	12.2704	12.2773	12.2719	12.2649	12.2723	12.275
	/K=1.1	15.2935	15.2878	15.2938	15.3005	15.295	15.287	15.2954	15.2976
Standard Error:4.67E-0.5									

The results presented in Tables 4,5 and 6 show that the differences in the computed option prices are of a very small magnitude. This was expected since the prices produced using the lognormal distribution as an approximation for the distribution of the sum of n lognormal variables, were already close to the MC estimates. Even though the price estimates seem to be less accurate that the normal ones[1] the differences are in the third decimal digit where the Monte Carlo error begins to become significant. Furthermore, because of the methodology used to produce the Monte Carlo prices for Basket options, as deviations form the "lognormally" distributed option prices, they were expected to be very close to the latter.

7.7 CONCLUSIONS

Closed form expression for Basket call and put options are provided in this chapter, along with expressions for the risk management parameters. The results have been generated both under the assumption that the sum of lognormal variables is also a lognormal variable and under the assumption that the distribution of the sum of lognormal variables can be described by am Edgeworth series expansion. Monte Carlo simulation validates the results.

[1] The highest percentage deviation between the Monte Carlo estimates and the analytically computed option prices in this case was less than 0.8%

8 CHAPTER EIGHT: SUMMARY, CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

8.1 Summary

My Ph.D. thesis has as a goal to attempt to exploit the uses of implied distributions and of implied information in general.

Implied models and generally the utilisation of implied information is one of the main current research topics. Non-implied models collect their information mainly from statistical properties of the time-series data of the underlying asset or from purely theoretical arguments. The innovation that implied models introduced is that information embedded in option prices is used directly without having to be filtered through the underlying asset's properties.

This directness is the informational gain that implied models are offering. Furthermore, option implied risk neutral distributions are forward looking while time-series estimated distributions are based on past data. Thus they are able to incorporate a wide range of future eventualities that simply are not captured using historical data. They do not require a long historical time series in order to be estimated accurately and in addition they are able to reflect immediately a change in market sentiment. A sudden shift in beliefs due to a political announcements or economic news could be immediately captured in option prices and the implied PDF. Furthermore, they are capable of capturing the uncertainty inherent in financial markets that of "multiple scenarios". The shape of a potential distribution function will depend on market data across strike prices rather than on a mathematical function of the standard errors of an econometric regression model.

Option embedded information has been exploited by researchers and practitioners for purposes of direct recovery of the market sentiment for an additional reason. The use of implied methodologies is more advantageous over the alternative of simply using the futures markets to extract information concerning subsequent dates. Futures contracts merely reflect the market expectation of the future exchange rate, which corresponds to the mean of the risk neutral distribution of the exchange rate, not revealing any information on its variance or higher moments. These higher moments, though, contain information concerning the level of dispersion providing an indication of the uncertainty present in the market place.

Investors make better-informed judgements by comparing their own views of future underlying asset movements with the market consensus. Central Banks, on the other hand find implied modelling most appealing, since they can obtain a strong understanding on market's views, upon which they base their decision making.

However, the merits from the use of implied distributions are twofold. Firstly, as argued above, they have been shown to be able to accurately recover the market sentiment, which renders them a very useful tool for policy makers. The second use of implied models is the pricing of more exotic products consistently with their simple vanilla counterparts.

Implied models assume that observed options are priced correctly, not looking for arbitrage opportunities, and proceed for the valuation of the more complicated products. Achieving consistent pricing of exotic with vanilla options implies that when the simple call and put options are used to hedge the more complicated or OTC structure then even if the products included in the hedge may not be correctly priced, the hedge, however, should remain valid and possibly unaffected. This should happen because the exotic and the vanilla instruments are priced using the same model and set of information and they should be expected to deviate from the "correct" prices by a similar amount. In my Ph.D. thesis I examine and exploit the two main uses of implied methodologies in Mathematical Finance; recovery of the market sentiment, as depicted in option prices and consistent pricing of exotic and vanilla products.

8.2 Conclusions

Chapter 5 examines whether the use of an implied distribution belonging to the parametric family of mixture of two lognormal distributions could have been used for the prediction of the exit of the British pound from the ERM in 1992. A combination of evidence presented in the chapter it can be strongly argued that, even though, the event of the pound's devaluation had not become foreseeable with absolute certainty until the very last day before it actually occurred, there were a number of indications in the market that could lead an investor safely enough, to that conclusion before the 15th of September. Therefore the Stirling pound's exit from the ERM did not catch by surprise the participants of the foreign exchange markets.

In Chapter 6 a new approach for pricing Asian options is developed that relaxes for the first time the restrictive assumption that the underlying asset's price distribution is lognormal. Both Geometric and Arithmetic Asian options are priced within a Bernoulli jump diffusion framework and analytic expressions for risk management parameters, ("Greeks") are derived. In order to determine the validity of the theoretical valuation results, I run Monte Carlo simulations so as to check whether option prices computed by the proposed model are in accordance with the ones postulated by the assumed distribution. In almost all the cases, theoretical option prices have less than one per cent deviation from their corresponding Monte Carlo estimates.

In chapter 7 a new methodology was developed for the pricing of Basket options. A Bernoulli jump diffusion framework was assumed for the law of motion of the assets comprising the asset and closed form expressions were derived for the prices of European call and put options. The expressions were derived under the assumption that the sum of lognormal variables is also a lognormal variable. This assumption was validated from the Monte Carlo experiments performed since the theoretically computed option prices were very close to the respective MC estimates. Closed form expressions for option prices sensitivities were also provided.

Closed form expressions were also derived under a more generalised assumption for the distribution of the sum of lognormal variables while still within the Bernoulli jump diffusion framework. The main conclusion drawn by comparing the two sets of results is that the sum of lognormal variables can be adequately described by another lognormal distribution for the purpose of Basket option pricing. The use of the Edgeworth series methodology does not significantly alter the "lognormally "computed option prices.

8.3 Suggestions for further research

This thesis examines the two most important uses of implied methodologies; the recovery of the market sentiment and the pricing of exotic and vanilla products consistent on the same set of information. The findings presented can be extended in a number of ways.

The validity of the Bernoulli jump diffusion model that is used in chapters 6 and 7 can be questioned since it only allows for only one jump for the underlying asset during the life of the derivative. This is an inherent limitation of the model. This model feature makes it an in appropriate candidate for the modelling of longer maturity derivative products, since in a volatile market one would expect more than one jumps to occur while the option is traded.

Currently the dominant model used in the literature for jump modelling is the jump diffusion model with Poisson distributed jump component, which assumes that the number of jumps allows for the underlying asset is proportional to the time period between now and the maturity of the option. This is clearly a more realistic assumption, however, this model is analytically intractable and therefore no closed form solutions can be derived for exotic products.

In order to mitigate the effect of the assumption that at most one jump is allowed during the life of the option, it is possible to extend the methodology presented in this thesis and use a model that allows two or three jumps in the same time interval. Each jump would have a probability associated with it, which would be implicitly derived from the market. Implicit recovery of the model parameters would be possible because, for example, a stochastic process that allows for at most three jumps between now and option maturity is consistent with a mixture of four lognormal distributions, in the same way that the stochastic process presented in this thesis is consistent with a mixture of two lognormal distributions. The parameter estimation procedure for a mixture of lognormals distribution has been described by Melick and Thomas (1996) who use a mixture of three lognormals for implied distribution recovery purposes. The valuation formulae though under the more general stochastic processes would be significantly more complex being, however, in analytical form.

The development of such a model would provide a generalisation of the current methodology. Both the current methodology and the extensions suggested are nested within the Poisson stochastic process assumption. Thus, if further research were undertaken towards this direction then we would come considerably closer to analytically pricing exotic derivatives within the widely accepted framework of Poisson jump diffusions.

In this thesis closed form expressions are provided for the valuation and hedging parameters of Asian and Basket options. The framework that these exotic products are priced within, can be used for the pricing of alternative products as well that are of great interest to financial market participants. Namely, further research can be undertaken in order to deal with the valuation problem of lookback options. Lookback options are options that pay the difference between the maximum price obtained by the underlying asset during the life of the option and the strike price of the option or between a fixed strike price and the minimum value obtained by the asset. This category of lookback options are called fixed strike options. There are other varieties in the same family of products, such as the floating strike lookbacks.

The valuation of these products presents a number of difficulties because of the nature of the stochastic problem that has to be solved in order to obtain the final solution. As it was mentioned the price of a lookback option depends on the maximum (call) or the minimum (put) obtained by the underlying asset over a pre-specified period. The joint density of the maximum of a Geometric Brownian motion and its location has been studied by Shepp(1979). Goldman, Sosin and Shepp(1979), Goldman, Sosin and Gatto (1978) and Conze and Viswanathan(1991) have studied the pricing problem of lookback options within a Geometric Brownian Motion framework and closed form expressions have been provided.

As one would expect, the extension of the pricing methodology to Bernoulli jump diffusion framework presents additional difficulties. In order to facilitate the computation of the option price it will be again necessary to condition on the time of the jump occurrence. However, even after the conditioning we are left with a five dimensional integral that has to be numerically evaluated since it contains terms of the form: $e^{x^2}\Phi(ax+b)$ rendering the analytical computation of the integral impossible. Furthermore, lookback options are very sensitive to the frequency that the underlying price is observed in order for the maximum or the minimum to be determined. This is bound to complicate the evaluation of the quintuple integral for the derivation of the option price. It would be of great interest if a way were developed to price these options within the Bernoulli jump diffusion framework that overcame the previously mentioned difficulties.

The use of the Bernoulli jump diffusion framework could also be further extended for the pricing of Barrier options. Barrier options are similar to European options with the important difference that they get activated or cancelled if the underlying asset hits a certain barrier during the life of the option. Because of this cancellation feature, or the conditional existence feature, they are cheaper than the corresponding European options, something that makes then very attractive to investors that are willing to bear this extra risk for a smaller up front premium.

The valuation of Barrier options has attracted a great number of researchers because of their extensive use by practitioners. Boyle and Tian (1999) price Barrier options under the CEV (constant elasticity of variance) process, Geman and Yor (1996) use a probabilistic approach to price double-barrier options and Zhou (1999) use non-continuous paths fro the underlying asset to price a barrier option, to mention only some of the most recent research papers devoted to these derivatives products. Closed form solutions have been derived for these instruments under the GBM assumption and various methods have been suggested for their valuation under more flexible assumptions. However, it would be of great interest to develop a pricing methodology for these instruments under the Bernoulli jump diffusion framework suggested in this thesis, since then it would be possible to consistently price these contracts with the vanilla options traded in the market. Furthermore, due to the analytical tractability of the latter framework it could be possible to provide analytical valuation formulae both for the valuation and the "Greeks" of the option.

The main difficulty arises from the fact the variable that represents the time that the underlying asset hits the barrier is a stopping time that cannot easily be modelled within a jump diffusion framework. However, Zhou (1999) has made some progress in this direction and the results presented in that paper can possibly be used as a guide for the methodology to be developed under the simpler Bernoulli jump diffusion framework.

The usefulness of implied distributions has been repetitively argued in the Finance literature (see Dumas Fleming and Whaley (1998), Gemmil and Saflekos (2000)). However, as it has been extensively argued in the text, in the author's opinion the techniques devised for determining the usefulness of implied distributions have been focusing in the wrong direction. Given the fact that the main advantage of using implied methodologies is the consistent pricing of vanilla and exotic products, the foundations for the ultimate test of the applicability of implied approaches is set in this thesis.

Assuming an implied stochastic process for the underlying asset and pricing exotic derivatives consistently with that process, the performance of a hedging strategy can be examined. The hedging parameters shall be estimated from the implied model using the set of information that the vanilla options provide. Then, if implied distributions were to be used for the consistent pricing of exotic and vanilla options the hedging performance of these strategies should be superior to that of a one more naïvely designed. In order to assess the performance, though, trading data on exotic options should be available. Thus, one would need to have access either to OTC data or alternatively use the very newly introduced Asian style products traded in some exchanges. One would have to be then extremely cautious in interpreting the results, since possible liquidity issues could affect the results. Furthermore, a study would have to be carried out to more than one different markets and products, in order to determine whether any results drawn, are market or product specific or they apply in general.

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